

TWISTED K -THEORY AND OBSTRUCTIONS AGAINST POSITIVE SCALAR CURVATURE METRICS

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ABSTRACT. Motivated by the index obstruction $\theta(M)$ to positive scalar curvature metrics as defined in the untwisted case by Rosenberg and in the twisted case by Stolz, we develop twisted K -theory with coefficients in a C^* -algebra A in terms of twisted Hilbert A -module bundles. We then decompose $\theta(M)$ as a pairing of a twisted K -homology with a twisted K -theory class and prove that $\theta(M)$ does not vanish if M is an orientable enlargeable manifold with spin universal cover, where the covers in the definition of enlargeability may have infinite numbers of leaves.

1. INTRODUCTION

In [27] Rosenberg constructed an index obstruction $\alpha(M) \in KO_n(C_{\mathbb{R}}^*(\pi_1(M)))$ for closed spin manifolds M of dimension n , which vanishes if M admits a metric of positive scalar curvature by the Lichnerowicz-Schrödinger-Weitzenböck formula. It takes values in the K -theory of the (maximal or reduced) real group C^* -algebra associated to the fundamental group and relies on the existence of a spin structure on M . The Gromov-Lawson-Rosenberg conjecture states that vanishing of $\alpha(M)$ implies the existence of a positive scalar curvature metric for M with $\dim(M) \geq 5$. It was proven to be true in the simply-connected case by Stolz [32], but is false in general as was shown by Schick [29].

Stolz also generalized this invariant to the case, where M itself may not be spin, but its universal cover \tilde{M} still is [31], [28, theorem 1.7]. He introduced the notion of supergroups, which are $\mathbb{Z}/2\mathbb{Z}$ -extensions of ordinary groups that additionally come equipped with a grading homomorphism. The index invariant $\theta(M) \in KO(C^*\gamma)$ takes values in the K -theory of a real C^* -algebra associated to a *twisted* version of the fundamental group accounting for the missing spin structure on M . Whereas $\alpha(M)$ in the case of spin manifolds was easily expressed as the pairing of the Dirac class $[D] \in KO_n(M)$ with the KO -theory class $[\mathcal{V}] \in KO_0(C(M, C^*\pi))$ of the Mishchenko-Fomenko bundle, $\theta(M)$ seems to lack this decomposition, since there is no Dirac operator on M .

Nevertheless, there is a K -orientation class in *twisted* K -homology [9, 2, 17]. We will switch to complex K -theory for convenience. Ordinary twists of K -theory are then classified by elements of $H^3(M, \mathbb{Z})$. In particular, the orientation twist is given by the torsion element $W_3(M) = \beta(w_2(M))$, where $\beta: H^2(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^3(M, \mathbb{Z})$ is the Bockstein homomorphism. Geometrically they may be represented by bundle gerbes developed by Murray [22]. These will be reviewed in section 2.1. There is a geometric description of twisted K -theory with torsion twist by modules over bundle gerbes contained in [6]. Most features of index theory are preserved in the

twisted case: Murray and Singer proved the analogue of the Atiyah-Singer index theorem in this context [24] and Carey and Wang proved the Thom isomorphism [8] (see also [9]).

Motivated by these results, the present paper develops twisted K -theory with coefficients in a unital C^* -algebra A , where the twists arise from a projective unitary action of a Lie group Γ on A , i.e. a group homomorphism $\tau: \Gamma \rightarrow U(A)/U(1)$, together with a principal Γ -bundle P over M (for more general twists see [26]). Let $\mathcal{A} = P \times_\tau A$. The algebra $C(M, \mathcal{A})$ of continuous sections is a C^* -algebra. Ordinary bundles of Hilbert A -modules represent classes in $K_0(C(M, A))$, where $C(M, A)$ is the algebra of continuous A -valued functions on M . Guided by the definition of bundle gerbe modules we define twisted Hilbert A -module bundles, which are ordinary Hilbert A -module bundles $E \rightarrow P$ that are “equivariant up to a tensor product with a line bundle”. The concept of twisted connections on these bundles is a straightforward generalization of ordinary A -linear connections. We prove an isomorphism of the corresponding Grothendieck group $K_{\mathcal{A}}^0(M)$ with $K_0(C(M, \mathcal{A}))$ and develop the complete framework of Mishchenko-Fomenko index theory in this twisted case:

There is a Chern character $\text{ch}_Q: K_{\mathcal{A}}^0(M) \rightarrow H^{\text{even}}(M, K_0(A) \otimes \mathbb{R})$ depending on a choice of trivialization Q . If A comes equipped with a trace τ it induces another Chern character $\text{ch}_\tau: K_{\mathcal{A}}^0(M) \rightarrow H^{\text{even}}(M, \mathbb{R})$ and both are connected via the notion of dimension induced by τ . Generalized projective Dirac operators as defined in section 3.1.4 provide a useful description of classes in twisted K -homology as close to the untwisted case as possible. After countertwisting with an appropriate twisted Hilbert A -module bundle they descend to an ordinary A -linear first-order elliptic differential operators, which have an index in $K_0(A)$ (see 3.33 and 3.36).

We then proceed to identify the (complex version of the) twisted index obstruction of Stolz as the pairing of a twisted K -homology with a twisted K -theory class:

$$\theta^{\text{max}}(M) = \text{ind}(D_+^{\mathcal{V}_{\text{max}}, Q_{\text{spin}}}) = [\mathcal{V}_{\text{max}}^{S, Q_1}] \otimes_{C(M, \mathcal{K})} [D^{S, Q_2}] .$$

This is more than just a reformulation of the work by Stolz: Twisted Hilbert A -module bundles are close enough to actual bundles to easily transfer proofs that worked in the untwisted case to the twisted setup. For example, for a bundle gerbe module there still is an analogue of the frame bundle as we prove in lemma 4.7. Parallel transport along a twisted connection is defined and yields a *projective* holonomy representation in the case of flat connections (see section 4.1).

To give just one example of the power of this technique, we prove that $\theta(M)$ does not vanish for orientable enlargeable manifolds as defined in 5.1 with spin universal cover and thereby enhance a result by Hanke and Schick [13, 14]:

Theorem 5.7. *Let M be a closed compact smooth orientable even-dimensional manifold with $\dim(M) \geq 3$ and \widetilde{M} spin that is enlargeable in the sense of definition 5.1. Then we have*

$$\theta^{\text{max}}(M) \neq 0 .$$

Gromov and Lawson showed that enlargeable manifolds M do not allow a metric of positive scalar curvature. However, they only worked with *finite* covers in the definition of enlargeability [12]. In contrast to their definition we allow the covers $\widetilde{M} \rightarrow M$ to be *non-compact* as in [14]. In this generality transfer arguments working for finite covers fail! Twisted Hilbert A -module bundles, however, provide

a way to circumvent this problem. Moreover, this result is independent of the injectivity of the twisted Baum-Connes map. This should convince the reader that twisted K -theory as presented in this paper provides the right setup to treat index obstructions in the twisted case.

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2. PRELIMINARIES

2.1. Bundle gerbes. Throughout the paper, M will denote a smooth compact closed orientable manifold and A will be a unital C^* -algebra if not stated otherwise. Abelian (S^1) -bundle gerbes were introduced by Murray in [22, 23], where they appeared as a geometric realization of classes in third cohomology with integer coefficients in analogy to line bundles which provide a geometric model for classes in $H^2(X, \mathbb{Z})$ for a topological space X . In this section we will review first their general definition and the special case of lifting bundle gerbes. We will discuss their basic properties and define trivializations and the associated Dixmier-Douady-class.

Definition 2.1. Let M be a smooth manifold and let $Y \rightarrow M$ be a surjective submersion. A line bundle $L \rightarrow Y^{[2]}$ will be called a *bundle gerbe* if there exists a multiplication over $Y^{[3]}$, i.e. an isomorphism of line bundles

$$\mu: \pi_{12}^* L \otimes \pi_{23}^* L \longrightarrow \pi_{13}^* L,$$

where $\pi_{ij}: Y^{[3]} \rightarrow Y^{[2]}$ denotes the canonical projections to the fiber product of the i th and j th factor, and such that over $Y^{[4]}$ the following diagram commutes

$$\begin{array}{ccc} (\pi_{12}^* L \otimes \pi_{23}^* L) \otimes \pi_{34}^* L & \xlongequal{\quad} & \pi_{12}^* L \otimes (\pi_{23}^* L \otimes \pi_{34}^* L) \\ \downarrow \mu \otimes \text{id} & & \downarrow \text{id} \otimes \mu \\ \pi_{13}^* L \otimes \pi_{34}^* L & & \pi_{12}^* L \otimes \pi_{24}^* L \\ \searrow \mu & & \swarrow \mu \\ & \pi_{14}^* L & \end{array}$$

Example 2.2. Let $Q \rightarrow Y$ be a line bundle. Then the canonical isomorphism $Q \otimes Q^* \rightarrow \underline{\mathbb{C}}$, where the latter denotes the trivial line bundle turns

$$\pi_1^* Q \otimes \pi_2^* Q^* \rightarrow Y^{[2]}$$

into a bundle gerbe, which is called the *trivial bundle gerbe* δQ . If L is a bundle gerbe, then L^* inherits a bundle gerbe multiplication from L . L^* is called the *dual bundle gerbe*.

Example 2.3. One of the main examples of bundle gerbes arises from central S^1 -extensions of Lie groups. Let

$$1 \rightarrow S^1 \rightarrow \widehat{\Gamma} \xrightarrow{q} \Gamma \rightarrow 1$$

be such a central S^1 -extension and let $P \rightarrow M$ be a principal Γ -bundle. Then there is a canonical map $\kappa: P^{[2]} \rightarrow \Gamma$ mapping a pair of points (p_1, p_2) in the same fiber over M to the (unique) group element connecting the two, i.e. to g_{12} with $p_1 g_{12} = p_2$. The *lifting bundle gerbe* L is now associated to the pullback $\widehat{L} = \kappa^* \widehat{\Gamma} \rightarrow M$, where the latter is a principal S^1 -bundle. If we denote points in L_{p_1, p_2} by $[\widehat{g}_{12}, \lambda]$ with $\widehat{g}_{12} \in \widehat{\Gamma}$ and $\lambda \in \mathbb{C}$ (we drop the points p_1, p_2 from the notation) then μ is given by

$$\mu([\widehat{g}_{12}, \lambda] \otimes [\widehat{g}_{23}, \lambda']) = [\widehat{g}_{12} \widehat{g}_{23}, \lambda \lambda'] .$$

Example 2.4. We are going to specialize example 2.3 even further and consider an extension

$$1 \rightarrow S^1 \rightarrow \widehat{\pi} \xrightarrow{q} \pi \rightarrow 1 ,$$

in which $\widehat{\pi}$ and π are *discrete* groups. As such it is classified by a cocycle $c_{\widehat{\pi}}$ representing an element $[c_{\widehat{\pi}}] \in H_{\text{gr}}^2(\pi, S^1)$ in the second group cohomology. Twisting the multiplication with $c_{\widehat{\pi}}$ we can identify $\widehat{\pi}$ with $\pi \times S^1$, such that

$$(g_1, z_1) \cdot (g_2, z_2) = (g_1 g_2, z_1 z_2 c_{\widehat{\pi}}(g_1, g_2)) .$$

A principal π -bundle over M now corresponds to a cover $\bar{M} \rightarrow M$ (which may be non-connected) and $L \rightarrow \bar{M}^{[2]}$ is in fact trivial *as a line bundle*. Nevertheless, the bundle gerbe multiplication still gets twisted, i.e. if (g_{12}, λ) and (g_{23}, λ') are two points in $\bar{M}^{[2]} \times S^1$ (with a notation similar to example 2.3), then

$$\mu((g_{12}, \lambda), (g_{23}, \lambda')) = (g_{12} g_{23}, c_{\widehat{\pi}}(g_{12}, g_{23}) \lambda \lambda') .$$

Definition 2.5. A lifting bundle gerbe $L \rightarrow \bar{M}^{[2]}$ of the form described in example 2.4 will be called *covering bundle gerbe*.

Example 2.6. Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes. Denote by $\text{pr}_i: (Y_1 \times_M Y_2)^{[2]} \rightarrow Y_i^{[2]}$ the projections to both of the factors, then the *exterior tensor product*

$$L = L_1 \boxtimes L_2 = \text{pr}_1^* L_1 \otimes \text{pr}_2^* L_2$$

is again a bundle gerbe over $(Y_1 \times Y_2)^{[2]}$ in a canonical way. In particular, if L_i is a lifting bundle gerbe associated to a central extension

$$1 \rightarrow S^1 \rightarrow \widehat{\Gamma}_i \rightarrow \Gamma_i \rightarrow 1$$

and a principal Γ_i -bundle P_i , then we can form

$$\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2 = (\widehat{\Gamma}_1 \times \widehat{\Gamma}_2) / S^1$$

where S^1 acts on $\widehat{\Gamma}_1 \times \widehat{\Gamma}_2$ with respect to the anti-diagonal action $z \cdot (g_1, g_2) = (z g_1, \bar{z} g_2)$, and $L_1 \boxtimes L_2$ is the lifting bundle gerbe for the central S^1 -extension

$$1 \rightarrow S^1 \rightarrow \widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2 \rightarrow \Gamma_1 \times \Gamma_2 \rightarrow 1$$

and the principal $\Gamma_1 \times \Gamma_2$ -bundle $P_1 \times_M P_2$.

Remark 2.7. Let $\Delta: Y \rightarrow Y^{[2]}$; $y \mapsto (y, y)$ be the diagonal map and let L be a bundle gerbe. Due to the multiplication map μ , the pullback bundle Δ^*L has a canonical trivialization given by

$$(1) \quad \underline{\mathbb{C}} \rightarrow \Delta^*L^* \otimes \Delta^*L \rightarrow \Delta^*L^* \otimes \Delta^*L \otimes \Delta^*L \rightarrow \Delta^*L ,$$

where the first map is the canonical isomorphism, the second is $\text{id} \otimes \mu^{-1}$ and the last is induced by the canonical pairing on the first two factors.

2.1.1. *The Dixmier-Douady class.* Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. As an analogue of the first Chern class for complex line bundles, there is a class $dd(L) \in H^3(M, \mathbb{Z}) \cong \check{H}^2(M, S^1)$ associated to L , which is defined as follows. Choose a (good) cover $\bigcup_i U_i \supset M$ and sections $\sigma_i: U_i \rightarrow Y$ of the projection map $Y \rightarrow M$. Note that (σ_i, σ_j) maps $U_{ij} = U_i \cap U_j$ to $Y^{[2]}$ and set $L_{ij} = (\sigma_i, \sigma_j)^*L$. Choose trivializations $\kappa_{ij}: U_{ij} \times S^1 \rightarrow L_{ij}$, which exist by contractibility of U_{ij} . Over the triple intersection U_{ijk} there are now two trivializations of L_{ik} : κ_{ik} as well as $\mu(\kappa_{ij} \otimes \kappa_{jk})$, where μ again denotes the bundle gerbe multiplication. They both differ by a map $\omega_{ijk}: U_{ijk} \rightarrow S^1$ that is:

$$\mu(\kappa_{ij} \otimes \kappa_{jk}) = \omega_{ijk} \kappa_{ik} .$$

The careful analysis given in [22] shows that ω_{ijk} is a Čech 2-cocycle that does not depend on all the choices up to coboundaries and therefore provides the class advertised in the beginning.

The following theorem proven in [22] summarizes the most important properties of $dd(L)$.

Theorem 2.8. *Let $L \rightarrow Y^{[2]}$ and $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be bundle gerbes. The Dixmier-Douady class has the following properties:*

- a) $dd(L) = 0$ if and only if L is isomorphic over $Y^{[2]}$ to a trivial bundle gerbe. In particular, if L is a lifting bundle gerbe over $P^{[2]}$ for the short exact sequence

$$1 \rightarrow S^1 \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1$$

and a principal Γ -bundle P , then the latter lifts to a principal $\hat{\Gamma}$ -bundle if and only if $dd(L)$ vanishes.

- b) $dd(L_1 \boxtimes L_2) = dd(L_1) + dd(L_2)$. □

- c) $dd(L^*) = -dd(L)$.

In view of part a) of the above theorem, we state the following definition from [22]:

Definition 2.9. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. L is called *trivial* if there is a line bundle $Q \rightarrow Y$ and an isomorphism of bundle gerbes $L \rightarrow \delta Q$. A particular choice of Q and of the isomorphism $L \rightarrow \delta Q$ is called a *trivialization* of L .

2.1.2. *Connections on bundle gerbes.* Connections on bundle gerbes were already treated by Murray in [22]. Their definition is straightforward and one of the main advantages of bundle gerbes over other gerbe-like structures given by local data is that the former live in a differential geometric setting, such that notions like connection, parallel transport and curvature transfer quite naturally. Explicit formulas for connections on lifting bundle gerbes in terms of Lie algebra splits were given by Gomi in [11]. We will mostly stick to the case where the central extension

$$1 \rightarrow S^1 \rightarrow \hat{\Gamma} \rightarrow \Gamma \rightarrow 1$$

is itself flat as a principal S^1 -bundle. In this case, L carries a canonical flat connection.

Definition 2.10. Let L be a bundle gerbe, \widehat{L} its principal S^1 -bundle. A covariant derivative $\nabla^L: \Omega^0(L) \rightarrow \Omega^1(L)$ on L is called a *bundle gerbe connection* if the multiplication isomorphism

$$\mu: \pi_{12}^* L \otimes \pi_{23}^* L \rightarrow \pi_{13}^* L$$

pulls it back to the canonical connection on the tensor product, i.e.

$$(2) \quad \mu^* \pi_{13}^* \nabla^L = \pi_{12}^* \nabla^L \otimes 1 + 1 \otimes \pi_{23}^* \nabla^L .$$

Alternatively, we could describe ∇^L giving the *connection form* $\theta_L \in \Omega^1(\widehat{L}, i\mathbb{R})$. Aside from the conditions of equivariance and reproduction of generators of fundamental vector fields, θ_L has to satisfy

$$(3) \quad \mu^* \pi_{13}^* \theta_L = \pi_{12}^* \theta_L + \pi_{23}^* \theta_L ,$$

which is just the replacement of equation (2), where μ denotes the induced map on the principal S^1 -bundle.

Bundle gerbe connections were shown to exist in general in [22]. If L is a lifting bundle gerbe associated to a principal Γ -bundle and a short exact sequence

$$1 \rightarrow S^1 \rightarrow \widehat{\Gamma} \xrightarrow{q} \Gamma \rightarrow 1 ,$$

then we can consider the Maurer-Cartan forms $\mu_{\widehat{\Gamma}}$ and μ_{Γ} on $\widehat{\Gamma}$ and Γ respectively. Let $\widehat{\mathfrak{g}}$ be the Lie algebra of $\widehat{\Gamma}$, \mathfrak{g} that of Γ . Assume there exists a Lie algebra split $\sigma: \widehat{\mathfrak{g}} \rightarrow \mathfrak{g}$ of the short exact sequence

$$0 \rightarrow i\mathbb{R} \rightarrow \widehat{\mathfrak{g}} \xrightarrow{q_*} \mathfrak{g} \rightarrow 0 .$$

The 1-form $\nu_{\sigma} \in \Omega^1(\widehat{\Gamma}, i\mathbb{R})$

$$\nu_{\sigma} = \mu_{\widehat{\Gamma}} - \sigma \circ q^* \mu_{\Gamma} .$$

indeed takes values in $i\mathbb{R}$ due to the split property of σ . Its pullback via the characteristic map $\widehat{\kappa}: \widehat{L} \rightarrow \widehat{\Gamma}$ yields a bundle gerbe connection on L since σ commutes with the adjoint action of $\widehat{\Gamma}$ on $\widehat{\mathfrak{g}}$. This is a special case of the results in [11]. The existence of a Lie algebra split is quite a restriction on the extensions, even though it is satisfied in all applications we have in mind.

Theorem 2.11. *Let $\widehat{\Gamma}$ be a connected Lie group, which is a central S^1 -extension of another connected Lie group Γ . Then the following are equivalent:*

a) *The associated short exact sequence of Lie algebras*

$$(4) \quad 0 \rightarrow i\mathbb{R} \rightarrow \widehat{\mathfrak{g}} \xrightarrow{q_*} \mathfrak{g} \rightarrow 0 .$$

splits in the category of Lie algebras.

b) *$\widehat{\Gamma}$ is associated to the universal cover $\widetilde{\Gamma} \rightarrow \Gamma$, i.e. there is a group homomorphism $\rho: \pi_1(\Gamma) \rightarrow S^1$ such that $\widehat{\Gamma} = \widetilde{\Gamma} \times_{\rho} S^1$.*

c) *There is a section $\sigma: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ such that the curvature of the connection form ν_{σ} vanishes.*

Proof. If (4) splits in the category of Lie algebras, then there is a Lie algebra isomorphism $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus i\mathbb{R}$ and we can identify ν_σ with $\text{pr}_{i\mathbb{R}} \circ \mu_{\widehat{\mathfrak{g}}}$. Therefore by the Maurer-Cartan equation

$$d\nu_\sigma = \text{pr}_{i\mathbb{R}} \circ d\mu_{\widehat{\mathfrak{g}}} = -\text{pr}_{i\mathbb{R}} \circ [\mu_{\widehat{\mathfrak{g}}}, \mu_{\widehat{\mathfrak{g}}}] = 0,$$

since $i\mathbb{R} \subset \widehat{\mathfrak{g}}$ is a central abelian subalgebra. If there is a section $\sigma: \mathfrak{g} \rightarrow \widehat{\mathfrak{g}}$ such that $d\nu_\sigma = 0$, then this equation evaluated on elements $\sigma(X), \sigma(Y) \in \widehat{\mathfrak{g}}$ yields

$$[\sigma(X), \sigma(Y)] - \sigma([X, Y]) = 0$$

showing that σ is indeed a Lie algebra split. The equivalence of **b)** and **c)** is clear, because a principal bundle is flat if and only if it reduces to a cover of the base manifold. In particular this is true for the principal S^1 -bundle $\widehat{\Gamma} \rightarrow \Gamma$ (see also [25] for the infinite dimensional case). \square

Definition 2.12. A central extension that satisfies one of the conditions of the previous lemma will be called *flat*.

Remark 2.13. Let L be a bundle gerbe with bundle gerbe connection ∇^L , denote the diagonal embedding $Y \rightarrow Y^{[2]}$ by Δ and the pullback connection on Δ^*L by $\Delta^*\nabla^L$. Let $(\Delta^*\nabla^L)^*$ be the dual connection, then the canonical isomorphism

$$\underline{\mathbb{C}} \rightarrow \Delta^*L^* \otimes \Delta^*L$$

identifies the canonical flat connection on $\underline{\mathbb{C}}$ with the tensor product connection $(\Delta^*\nabla^L)^* \otimes \text{id}_L + \text{id}_L^* \otimes \Delta^*\nabla^L$. Therefore the canonical isomorphism (1) induced by the bundle gerbe multiplication identifies $\Delta^*\nabla^L$ with the canonical flat connection on $\underline{\mathbb{C}}$.

2.1.3. Curvings. If $L \rightarrow Y^{[2]}$ is a bundle gerbe equipped with a connection ∇^L , then its curvature is a 2-form $\Omega_L \in \Omega^2(Y^{[2]})$. It has been shown by Murray in [22] that the following sequence forms a complex with vanishing cohomology for every $k \in \mathbb{N}$:

$$(5) \quad 0 \rightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(Y) \xrightarrow{\delta} \Omega^k(Y^{[2]}) \xrightarrow{\delta} \Omega^k(Y^{[3]}) \xrightarrow{\delta} \dots,$$

where the maps $\delta: \Omega^k(Y^{[m]}) \rightarrow \Omega^k(Y^{[m+1]})$ are given by

$$\delta(\omega) = \sum_{l=1}^{m+1} (-1)^l \bar{\pi}_l^* \omega$$

and $\bar{\pi}_l: Y^{[m+1]} \rightarrow Y^{[m]}$ is the projection, which leaves out the l th factor of $Y^{[m+1]}$. The same argument given in [22] can be used to obtain (5) taking coefficients in a bundle pulled back from M , that is: If $B \rightarrow M$ is a vector bundle, then

$$(6) \quad 0 \rightarrow \Omega^k(M, B) \xrightarrow{\pi^*} \Omega^k(Y, \pi_M^* B) \xrightarrow{\delta} \Omega^k(Y^{[2]}, \pi_M^* B) \xrightarrow{\delta} \dots$$

is an acyclic complex as well.

If ∇^L is a bundle gerbe connection with curvature $\Omega_L \in \Omega^2(Y^{[2]})$, then the compatibility with μ in (8) translates to $\delta(\Omega_L) = 0$. Thus there is $f \in \Omega^2(Y)$ such that $\delta(f) = \Omega_L$.

Definition 2.14. Let L be a bundle gerbe. Let ∇^L be a bundle gerbe connection with curvature $\Omega_L \in \Omega^2(Y^{[2]})$, then a choice of $f \in \Omega^2(Y)$ with $\delta(f) = \Omega_L$ will be called a *curving* of ∇^L .

Remark 2.15. Since d and δ commute and $d\Omega_L = 0$ there exists a closed 3-form $\omega \in \Omega^3(M)$ with $\pi^*\omega = df$. It represents a cohomology class $dd^{\mathbb{R}}(L) \in H^3(M, \mathbb{R})$, which coincides with the image of $dd(L)$ in real cohomology as was shown by Murray in [22]. It is easy to see that the condition $dd^{\mathbb{R}}(L) = 0$ is equivalent to the existence of a closed curving.

2.2. Bundle gerbe modules. The motivation for bundle gerbe modules arose from considering topological charges in string theory. Based on the observation that these should be described by twisted versions of vector bundles in the presence of certain anomalies and should represent classes in twisted K -theory, Bouwknegt, Carey, Mathai, Murray and Stevenson developed a geometric model for the latter in [6], which was refined by Carey and Wang in [8]. In case the twist represents a torsion class, this model parallels the description of ordinary K -theory by vector bundles by replacing them with modules over bundle gerbes.

Even though this paper will not deal with the physics behind this, we will later on exploit this approach to find a twisted replacement for the Mishchenko-Fomenko line bundle.

Definition 2.16. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. A finite dimensional vector bundle $F \rightarrow Y$ together with an isomorphism

$$\gamma: L \otimes \pi_2^* F \rightarrow \pi_1^* F$$

is called a *bundle gerbe module* for L if the following associativity diagram commutes:

$$\begin{array}{ccc} (\pi_{12}^* L \otimes \pi_{23}^* L) \otimes \pi_3^* F & \xlongequal{\quad} & \pi_{12}^* L \otimes (\pi_{23}^* L \otimes \pi_3^* F) \\ \downarrow \mu \otimes \text{id} & & \downarrow \text{id} \otimes \gamma \\ \pi_{13}^* L \otimes \pi_3^* F & & \pi_{12}^* L \otimes \pi_2^* F \\ & \searrow \gamma & \swarrow \gamma \\ & \pi_1^* F & \end{array}$$

We will call γ the *action of L on F* or the *twisting of F* .

Example 2.17. Let $P \rightarrow M$ be a principal Γ -bundle and let L be the lifting bundle gerbe associated to a central S^1 -extension

$$1 \rightarrow S^1 \rightarrow \widehat{\Gamma} \rightarrow \Gamma \rightarrow 1 .$$

Let $\rho: \widehat{\Gamma} \rightarrow U(n)$ be a unitary representation of $\widehat{\Gamma}$ with $\rho(z\widehat{g}) = z\rho(\widehat{g})$ for $z \in S^1$ and $\widehat{g} \in \widehat{\Gamma}$. Then

$$F = P \times \mathbb{C}^n$$

is a bundle gerbe module with the action of L given by

$$\gamma([\widehat{g}, \lambda] \otimes v) = \lambda \rho(\widehat{g}) v .$$

In the most important special case of this example $P = P_{SO}$ is the frame bundle of an orientable Riemannian manifold M and $L = L_{\text{spin}}$ is the lifting bundle gerbe associated to the central S^1 -extension

$$1 \rightarrow S^1 \rightarrow \text{Spin}^c(n) \rightarrow SO(n) \rightarrow 1 .$$

L_{spin} provides a replacement for missing spin^c structures on M . Its Dixmier-Douady-class is given by $dd(L_{\text{spin}}) = W_3(M) = \beta(w_2(M))$, where

$$\beta: H^2(M, \mathbb{Z}/2\mathbb{Z}) \rightarrow H^3(M, \mathbb{Z})$$

is the Bockstein homomorphism. In fact, trivializations of this gerbe are in 1 : 1-correspondence with spin^c structures on M . L_{spin} has been considered for example in [24, 20, 6, 8].

The irreducible representation of $\text{Spin}^c(n)$ on \mathbb{C}^N provides via the above construction an example of a bundle gerbe module, which will be called the *spinor module*.

Example 2.18. Now consider the setup given in example 2.4, that is let

$$1 \rightarrow S^1 \rightarrow \widehat{\pi} \xrightarrow{q} \pi \rightarrow 1 ,$$

be a central S^1 -extension of the discrete group π classified by a cocycle $c_{\widehat{\pi}} \in H_{\text{gr}}^2(\pi, S^1)$ and denote the corresponding covering bundle gerbe by $L_{\widehat{\pi}} \rightarrow \bar{M}^{[2]}$. Since $L_{\widehat{\pi}}$ is trivial as a vector bundle, a bundle gerbe module $F \rightarrow \bar{M}$ for $L_{\widehat{\pi}}$ consists of a vector bundle F over \bar{M} together with fiber isomorphisms

$$\gamma^g: F_{\bar{m}} \xrightarrow{\sim} F_{\bar{m}g^{-1}}$$

for $g \in \pi$, such that $\gamma^g \circ \gamma^h = c_{\widehat{\pi}}(g, h) \gamma^{gh}$, i.e. γ^g acts on the fibers of F like a projective representation of $(\widehat{\pi}, c_{\widehat{\pi}})$.

The correct notion of morphisms should respect the action of the bundle gerbe L .

Definition 2.19. Let F, F' be two bundle gerbe modules with respect to the same bundle gerbe L and denote the twistings by γ and γ' . A linear map $f: F \rightarrow F'$ will be called a *morphism of bundle gerbe modules* if the following diagram commutes:

$$\begin{array}{ccc} L \otimes \pi_2^* F & \xrightarrow{\gamma} & \pi_1^* F \\ \downarrow \text{id}_L \otimes \pi_2^* f & & \downarrow \pi_1^* f \\ L \otimes \pi_2^* F' & \xrightarrow{\gamma'} & \pi_1^* F' \end{array}$$

Let $\text{Hom}(F, F')$ be the homomorphism bundle of F and F' over Y and let $\pi_i: Y^{[2]} \rightarrow Y$ be the projection to the i th factor. The twistings γ and γ' yield an isomorphism:

$$\Psi: \pi_1^* \text{Hom}(F, F') \rightarrow \pi_2^* \text{Hom}(F, F')$$

defined as follows: Let $f: F_{y_1} \rightarrow F'_{y_1}$ be a linear map of the fibers at some point $y_1 \in Y$, then

$$\Psi(f): F_{y_2} \xrightarrow{\gamma^{-1}} L_{(y_2, y_1)} \otimes F_{y_1} \xrightarrow{\text{id}_L \otimes f} L_{(y_2, y_1)} \otimes F'_{y_1} \xrightarrow{\gamma'} F'_{y_2} .$$

If $\pi_{ij}: Y^{[3]} \rightarrow Y^{[2]}$ is the projection to the (i, j) th factor of $Y^{[2]}$ in $Y^{[3]}$ then the associativity of L implies that $\pi_{23}^* \Psi \circ \pi_{12}^* \Psi = \pi_{13}^* \Psi$. From this we deduce that $\text{Hom}(F, F')$ descends to a bundle $\text{hom}(F, F') \rightarrow M$, i.e.

$$\text{Hom}(F, F') \cong \pi^* \text{hom}(F, F') .$$

Global sections of $\text{hom}(F, F')$ correspond precisely to bundle gerbe morphisms from F to F' over Y . The punchline is that even though the modules over L are twisted,

their homomorphism bundles descend to honest vector bundles over M . In particular, the endomorphism bundle $\text{End}(F)$ descends to a bundle of matrix algebras $\text{end}(F)$ (see also [20]).

Example 2.20. If L is a lifting bundle gerbe and F is given by a representation $\rho: \hat{\Gamma} \rightarrow U(n)$ like in example 2.17, then ρ factors as a homomorphism $\bar{\rho}: \Gamma \rightarrow PU(n)$. $PU(n)$ acts via the adjoint action on $M_n(\mathbb{C})$ and if we denote the induced action of Γ by $\text{Ad}_{\bar{\rho}}$ we have

$$\text{end}(F) = P \times_{\text{Ad}_{\bar{\rho}}} M_n(\mathbb{C}) .$$

Definition 2.21. Like in [6] we define $K_L^0(M)$ to be the Grothendieck group of bundle gerbe modules for L .

2.2.1. *Connections on bundle gerbe modules.* Since the endomorphism bundle of a module F over some bundle gerbe L descends to a bundle over M , there is some hope to define a connection on F , which is compatible with a corresponding connection on L such that the curvature descends to a form on M , which would enable us, to define characteristic classes of F in terms of Chern-Weil theory. This is indeed possible as has already been observed in [6] and can be used to define a Chern character for bundle gerbe modules. If we denote the connection on L by ∇^L , it is straightforward to define the compatibility condition for the connection on F .

Definition 2.22. Let F be a bundle gerbe module for L . A connection ∇^F on F is called a *bundle gerbe module connection* if

$$(7) \quad \gamma^* \pi_1^* \nabla^F = \nabla^L \otimes \text{id} + \text{id} \otimes \pi_2^* \nabla^F .$$

If $\theta_L \in \Omega^1(\hat{L}, i\mathbb{R})$ is the connection form of ∇^L , $P_F \rightarrow Y$ is the frame bundle of F , n is its rank and $\eta_F \in \Omega^1(P_F, \mathfrak{u}(n))$ is the connection form for ∇^F , then (7) can be rephrased as follows:

$$(8) \quad \gamma^* \pi_1^* \eta_F = \theta_L + \pi_2^* \eta_F ,$$

where the embedding $i\mathbb{R} \rightarrow \mathfrak{u}(n)$ implicitly needed in (8) is induced by the canonical homomorphism $U(1) \rightarrow U(n)$. By a slight abuse of notation we identify the isomorphism γ with the map induced by it on the frame bundles.

Remark 2.23. The curvature form of ∇^F is an element $\Omega_F \in \Omega^2(Y, \pi^* \text{end}(F))$, which satisfies $\delta(\Omega_F) = \Omega_L$. Thus, with a choice of curving $f \in \Omega^2(Y)$ for ∇^L we get $\delta(\Omega_F - f) = 0$. From (6) we deduce that $\Omega_F - f = \pi^* \omega_f$ for a 2-form $\omega_f \in \Omega^2(M, \text{end}(F))$. In particular, $\Omega_F = \pi^* \omega_0$ (i.e. the curvature descends to a 2-form ω_0 on M) if ∇^L is a flat connection.

Example 2.24. Let F be the bundle gerbe module from example 2.17 for a flat central extension and let $\eta \in \Omega^1(P, \mathfrak{g})$ be a connection form on the principal Γ -bundle P with values in the Lie algebra \mathfrak{g} of Γ . The frame bundle of F is the trivial $U(n)$ -bundle over P . Denote its canonical projections by π_P and $\pi_{U(n)}$. Let $\mu_{U(n)}$ be the Maurer-Cartan form on $U(n)$. Then

$$\eta_F = \text{Ad}_{\pi_{U(n)}^{-1}} \rho_* \pi_P^* \eta + \pi_{U(n)}^* \mu_{U(n)}$$

yields a bundle gerbe module connection on F with respect to the canonical flat connection on the lifting bundle gerbe L . Let $\sigma \in \Gamma(F) = C^\infty(P, \mathbb{C}^n)$ be a section

of F over P , then the associated covariant derivative ∇^F acts via

$$\nabla^F(\sigma) = d\sigma + \rho_*(\eta) \cdot \sigma .$$

2.2.2. Trivializations and Descent. In case the Dixmier-Douady-class $dd(L)$ vanishes, bundle gerbe modules for L should be in 1 : 1-correspondence with vector bundles over the base space M , that is $E \rightarrow Y$ should descent to some vector bundle $\tilde{E} \rightarrow M$. In this section we will review this construction to check that it will also work for the twisted Hilbert A -module bundles defined later on.

Lemma 2.25. *Let $E \rightarrow Y$ be a locally trivial bundle with fiber V over the total space of a fibration $\pi: Y \rightarrow M$. Assume that there is a bundle isomorphism*

$$\phi: \pi_2^* E \xrightarrow{\cong} \pi_1^* E ,$$

where $\pi_i: Y^{[2]} \rightarrow Y$ denote the canonical projections. If the following associativity diagram over $Y^{[3]}$ commutes,

$$\begin{array}{ccc} \pi_3^* E & \xrightarrow{\pi_{23}^* \phi} & \pi_2^* E \\ & \searrow \pi_{13}^* \phi & \downarrow \pi_{12}^* \phi \\ & & \pi_1^* E \end{array}$$

then there is a bundle $\tilde{E} \rightarrow M$ with fiber V and an isomorphism $E \rightarrow \pi^* \tilde{E}$.

Proof. The proof is straightforward and we will only provide a sketch: Cover M by contractible sets $M \supset \bigcup_{i \in I} U_i$ and choose sections $\sigma_i: U_i \rightarrow Y$. Let $E_i = \sigma_i^* E$ and denote the maps induced by ϕ on the double intersections $U_{ij} = U_i \cap U_j$ by $\phi_{ij}: E_j \rightarrow E_i$. Now we can define

$$\tilde{E} = \coprod_{i \in I} E_i / \sim$$

where the equivalence relation is induced by the maps ϕ_{ij} . \square

We will apply the above lemma to bundle gerbe modules for the tensor product of two bundle gerbes L_1 and L_2 with $dd(L_1) = dd(L_2)$. Note that in this case $dd(L_1^* \boxtimes L_2) = -dd(L_1) + dd(L_2) = 0$, i.e. a trivialization of $L_1^* \boxtimes L_2$ exists.

Theorem 2.26. *Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes with $dd(L_1) = dd(L_2)$. Let $Q \rightarrow Y_1 \times_M Y_2$ be a trivialization of $L_1^* \boxtimes L_2 \rightarrow (Y_1 \times_M Y_2)^{[2]}$. If $F \rightarrow Y_1$ is a bundle gerbe module for L_1 and $\rho: Y_1 \times_M Y_2 \rightarrow Y_1$ denotes the canonical projection, then $\rho^* F \otimes Q$ descends to a bundle gerbe module*

$$Q(F) \rightarrow Y_2$$

for L_2 . If F' is another bundle gerbe module for L_1 and $\varphi: F \rightarrow F'$ is a twisted morphism, then there is a corresponding morphism $Q(\varphi): Q(F) \rightarrow Q(F')$ turning Q into a covariant functor.

Proof. In view of the last lemma we have to prove that $\pi_2^*(\rho^* F \otimes Q) \cong \pi_1^*(\rho^* F \otimes Q)$, where $\pi_i: Y_1^{[2]} \times_M Y_2 \rightarrow Y_1 \times_M Y_2$ denotes the projection to the i th Y_1 -factor. Moreover, we need to check that associativity holds and that the result carries an action of L_2 .

Since we have to distinguish a lot of different projection maps, we will give the needed isomorphisms on the fibers although it is possible to express all of them in terms of pullbacks via certain projections. Furthermore, we will always identify $(L_i)_{(y,y)}$ with \mathbb{C} using the canonical trivialization given in (1), which is compatible with the product operation. Since Q trivializes $L_1^* \boxtimes L_2$ we have the following isomorphism for $(y_1^i, y_2^i) \in Y_i^{[2]}$

$$(L_1^*)_{(y_1^1, y_2^1)} \otimes (L_2)_{(y_1^2, y_2^2)} \rightarrow Q_{(y_1^1, y_1^2)} \otimes (Q^*)_{(y_2^1, y_2^2)} .$$

In particular, we get

$$(L_1^*)_{(y_1^1, y_2^1)} \otimes Q_{(y_2^1, y^2)} \rightarrow Q_{(y_1^1, y^2)} .$$

by setting $y_1^2 = y_2^2 = y^2 \in Y_2$. From this, we can deduce the isomorphism $\pi_2^*(\rho^* F \otimes Q) \cong \pi_1^*(\rho^* F \otimes Q)$:

$$F_{y_2^1} \otimes Q_{(y_2^1, y^2)} \rightarrow (L_1)_{(y_1^1, y_2^1)} \otimes F_{y_2^1} \otimes (L_1^*)_{(y_1^1, y_2^1)} \otimes Q_{(y_2^1, y^2)} \rightarrow F_{y_1^1} \otimes Q_{(y_1^1, y^2)} .$$

The first map is the canonical identification $L_1 \otimes L_1^* \rightarrow \mathbb{C}$, the second is induced by the action of L_1 on F and the above map. Likewise, the other diagonal embedding $Y_1 \times_M Y_2^{[2]} \rightarrow (Y_1 \times_M Y_2)^{[2]}$ yields an isomorphism

$$(L_2)_{(y_1^2, y_2^2)} \otimes Q_{(y^1, y_2^2)} \rightarrow Q_{(y^1, y_1^2)}$$

for $(y_1^2, y_2^2) \in Y_2^{[2]}$ and $y^1 \in Y_1$. After tensoring with F this becomes the twisting map

$$(L_2)_{(y_1^2, y_2^2)} \otimes F_{y^1} \otimes Q_{(y^1, y_2^2)} \rightarrow F_{y^1} \otimes Q_{(y^1, y_1^2)} .$$

By the associativity of $L_1^* \boxtimes L_2$ and the fact that the above two maps are derived from a bundle gerbe isomorphism, this twisting commutes with the action that is used to define the descend bundle $Q(F) = \widetilde{\rho^* F \otimes Q}$. Moreover, this implies the associativity of the twisting as well as the associativity of the descend isomorphism. Thus, $Q(F)$ is a bundle gerbe module. If $\varphi: F \rightarrow F'$ is a twisted morphism, it commutes with the action L_1 and therefore yields a well-defined linear map $Q(\varphi): Q(F) \rightarrow Q(F')$. Since the action of L_2 on $Q(F)$ and $Q(F')$ only affects Q , $Q(\varphi)$ commutes with it, thus it is a twisted morphism. \square

As a special case of this construction we get

Corollary 2.27. *Let $L \rightarrow Y^{[2]}$ be a bundle gerbe with $dd(L) = 0$. Let $Q \rightarrow Y$ be a trivialization of L and $F \rightarrow Y$ be a bundle gerbe module for L , then $F \otimes Q^*$ descends to a vector bundle $Q^*(F) \rightarrow M$. Thus, Q^* is a natural equivalence between bundle gerbe modules for L and vector bundles over M .*

Proof. This follows directly from the above theorem if we choose $L_1 = L$ and the trivial bundle gerbe $M \times S^1$ over the identity fibration $M \rightarrow M$ for L_2 . \square

In the situation of theorem 2.26 sections of $Q(F) \rightarrow Y_2$ are identified via pullback with $\pi: Y_1 \times_M Y_2 \rightarrow Y_2$ with those sections σ of $\rho^* F \otimes Q \rightarrow Y_1 \times_M Y_2$, which satisfy

$$(9) \quad \phi \circ \pi_2^* \sigma = \pi_1^* \sigma ,$$

where $\pi_i: Y_1^{[2]} \times_M Y_2 \rightarrow Y_1 \times_M Y_2$ are the canonical projections and ϕ is the isomorphism $\pi_2^*(\rho^* F \otimes Q) \rightarrow \pi_1^*(\rho^* F \otimes Q)$. This can be used to channel connections through our trivialization Q as well.

Lemma 2.28. *Let $E \rightarrow Y$ be a smooth vector bundle over the total space of a smooth fibration $\pi: Y \rightarrow M$. Assume that there is a bundle isomorphism*

$$\phi: \pi_2^* E \xrightarrow{\cong} \pi_1^* E ,$$

such that the associativity of theorem 2.26 holds. If E carries a connection ∇^E with $\phi^ \pi_2^* \nabla^E = \pi_1^* \nabla^E$ then it descends to a connection $\nabla^{\tilde{E}}$ on \tilde{E} .*

Proof. Fix a point $(y_1, y_2) \in Y^{[2]}$ and a vector $(V_1, V_2) \in T_{(y_1, y_2)} Y^{[2]}$, i.e. $V_i \in T_{y_i} Y$ and $\pi_* V_1 = \pi_* V_2$. Let $\sigma: Y \rightarrow E$ be a section of E , which satisfies (9) and therefore corresponds to a pullback of some $\tilde{\sigma}: M \rightarrow \tilde{E}$. Now note that

$$\begin{aligned} \pi_1^* (\nabla_{V_1}^E \sigma) &= (\pi_1^* \nabla^E)_{(V_1, V_2)} \pi_1^* \sigma = (\pi_1^* \nabla^E)_{(V_1, V_2)} (\phi \circ \pi_2^* \sigma) \\ &= \phi \circ ((\pi_2^* \nabla^E)_{(V_1, V_2)} \pi_2^* \sigma) = \phi \circ \pi_2^* (\nabla_{V_2}^E \sigma) . \end{aligned}$$

Let $V_1 = 0$ and $V = V_2 \in \ker \pi_*$. Then it follows that $\nabla_V^E \sigma = 0$. Thus, for an arbitrary vector $V \in TY$ the value of $\nabla_V^E \sigma$ only depends on $\pi_* V \in T$ and not on the particular choice of lift to TY . Therefore $\nabla_W^E \sigma$ is well-defined for a vector field $W: M \rightarrow TM$ and $\nabla_W^E \sigma$ also satisfies (9). \square

Let $L_i \rightarrow P_i^{[2]}$ for $i \in \{1, 2\}$, F and Q be as in theorem 2.26. Let ∇^{L_i} be a connection on L_i . $L_1^* \boxtimes L_2$ inherits a connection $\nabla^{L_1^* \boxtimes L_2}$ from L_1 and L_2 . The isomorphism $L_1^* \boxtimes L_2 \cong \pi_1^* Q \otimes \pi_2^* Q^*$ together with its compatibility with the product equips Q with the structure of a rank 1-bundle gerbe module for $L_1^* \boxtimes L_2$. Let ∇^Q be a bundle gerbe module connection on Q for $\nabla^{L_1^* \boxtimes L_2}$.

Theorem 2.29. *Let L_i, F and Q be equipped with connections ∇^{L_i}, ∇^F and ∇^Q . Then $\nabla^{\rho^* F \otimes Q} = \rho^* \nabla^F \otimes \text{id} + \text{id} \otimes \nabla^Q$ descends to a bundle gerbe module connection $\nabla^{Q(F)}$ on $Q(F) \rightarrow Y_2$.*

Proof. Let $\pi_i^2: Y_1 \times_M Y_2^{[2]} \rightarrow Y_1 \times_M Y_2$, $\pi_j^1: Y_1^{[2]} \times Y_2 \rightarrow Y_1 \times_M Y_2$ be the canonical projections. By remark 2.13 we get two isomorphisms

$$\kappa_2: L_2 \otimes (\pi_2^2)^* Q \rightarrow (\pi_1^2)^* Q \quad , \quad \kappa_1: (\pi_2^1)^* Q \rightarrow (\pi_1^1)^* Q \otimes L_1 ,$$

where we have identified L_i with its pullback to shorten notation. Since ∇^Q is a bundle gerbe module connection we get the two identities

$$(10) \quad \kappa_2^{-1} \circ (\pi_1^2)^* \nabla^Q \circ \kappa_2 = \nabla^{L_2} \otimes \text{id}_Q + \text{id}_{L_2} \otimes (\pi_2^2)^* \nabla^Q ,$$

$$(11) \quad \kappa_1 \circ (\pi_2^1)^* \nabla^Q \circ \kappa_1^{-1} = \text{id}_Q \otimes \nabla^{L_1} + (\pi_1^1)^* \nabla^Q \otimes \text{id}_{L_2} ,$$

where κ_i acts on sections via pullback. Remember that $\rho^* F \otimes Q$ descends to a bundle over Y_2 via the descend data given by a consistent isomorphism

$$\psi: (\pi_2^1)^* (\rho^* F \otimes Q) \xrightarrow{\sim} (\pi_1^1)^* (\rho^* F \otimes Q)$$

composed from the twisting γ and κ_1 . We see from (11) that the canonical tensor product connection

$$\nabla^{\rho^* F \otimes Q} = \rho^* \nabla^F \otimes \text{id}_Q + \text{id}_{\rho^* F} \otimes \nabla^Q$$

satisfies

$$\psi^{-1} \circ (\pi_1^1)^* \nabla^{\rho^* F \otimes Q} \circ \psi = (\pi_2^1)^* \nabla^{\rho^* F \otimes Q}$$

and therefore defines a connection $\nabla^{Q(E)}$ on $Q(E) \rightarrow Y_2$ by lemma 2.28, which turns out to be a twisted connection by (10). \square

Remark 2.30. Let L_i for $i \in \{1, 2, 3\}$ be bundle gerbes over the spaces Y_i . Suppose we have two trivializations Q_1 of $L_1^* \boxtimes L_2$ and Q_2 of $L_2^* \boxtimes L_3$. As the notation above already suggested we can see Q_i as a generalized morphism from L_i to L_{i+1} . From this point of view we can ask if there is a notion of composition for trivializations. Indeed, there is such a construction, which is well-defined on *isomorphism classes* of trivializations. To see this, consider the exterior tensor product $Q_1 \boxtimes Q_2$. It is a line bundle over $Y_1 \times_M Y_2 \times_M Y_3$ that comes equipped with an isomorphism

$$\begin{aligned} (Q_1 \boxtimes Q_2)_{(y_1, y_2, y_3)} &\cong (Q_1 \boxtimes Q_2)_{(y_1, y'_2, y_3)} \otimes (L_2)_{(y_2, y'_2)}^* \otimes (L_2)_{(y_2, y'_2)} \\ &\cong (Q_1 \boxtimes Q_2)_{(y_1, y'_2, y_3)} \end{aligned}$$

Therefore it descends to a line bundle $Q_2 \circ Q_1$ over $Y_1 \times_M Y_3$ by lemma 2.25, which provides a trivialization of $L_1^* \boxtimes L_3$. Let ∇^{Q_i} be a bundle gerbe module connection on Q_i with respect to the action of $L_i^* \boxtimes L_{i+1}$. Using the same argument as in 2.29 we see that there is an induced connection $\nabla^{Q_2 \circ Q_1}$ on the composition as well.

Remark 2.31. Suppose L is a bundle gerbe with $dd(L) = 0$ and Q and Q' are two trivializations of L . Then it follows from

$$\pi_1^* Q \otimes \pi_2^* Q^* \cong \pi_1^* Q' \otimes \pi_2^* (Q')^* \Rightarrow \pi_1^* (Q \otimes (Q')^*) \cong \pi_2^* (Q \otimes (Q')^*)$$

that $Q \otimes (Q')^*$ descends to a line bundle ℓ over M , i.e. $Q \cong Q' \otimes \pi^* \ell$. Moreover, if ℓ is an arbitrary line bundle over M , it is easily checked that for a trivialization Q , $Q \otimes \pi^* \ell$ is another one. Now suppose that L_i are bundle gerbes as in remark 2.30 and let Q_{12} be a trivialization of $L_1^* \boxtimes L_3$. Let Q_1 and Q_2 be arbitrary trivializations of $L_1^* \boxtimes L_2$ and $L_2^* \boxtimes L_3$ respectively. In this situation there is a line bundle ℓ_{12} over M , such that $Q_{12} \cong (Q_2 \circ Q_1) \otimes \pi^* \ell_{12} \cong Q_2 \circ (Q_1 \otimes \pi^* \ell_{12})$. Therefore any trivialization Q_{12} can be decomposed into $Q_2 \circ Q_1$ for two trivializations Q_i .

2.3. Universal cover of principal bundles. This section contains some lemmata about universal covers of principal bundles. Most observations will be based on elementary covering theory. Consider first an extension G of the Lie group H by the Lie group A , i.e. a short exact sequence

$$1 \rightarrow A \longrightarrow G \xrightarrow{q} H \rightarrow 1 ,$$

such that $q: G \rightarrow H$ is a principal A -bundle.

Lemma 2.32. *Let $\pi_2: P_2 \rightarrow M$ be a principal H -bundle and $\pi_1: P_1 \rightarrow P_2$ be a principal A -bundle. If P_1 carries an action by G that is compatible with these structures in the sense that its restriction to A coincides with the given one and*

$$\pi_1(p \cdot g) = \pi_1(p) \cdot q(g)$$

for $p \in P_1$ and $g \in G$, then $\pi = \pi_2 \circ \pi_1: P_1 \rightarrow M$ is a principal G -bundle.

Proof. The proof is straightforward and we will omit it. \square

Let $\pi: P \rightarrow M$ be a principal Γ -bundle over M for a path connected Lie group Γ . Let $G \subset \pi_1(P)$ be a subgroup and denote by $\varrho: \bar{M} \rightarrow M$ the universal cover of M . Since $P \rightarrow M$ is a fibration we can consider the following excerpt of the homotopy long exact sequence

$$\pi_2(M) \longrightarrow \pi_1(\Gamma) \xrightarrow{\alpha} \pi_1(P) \xrightarrow{\pi_*} \pi_1(M) \rightarrow 1 .$$

The subgroup G classifies a cover $\bar{P} \rightarrow P$. Likewise denote by \bar{M} the cover of M classified by $\pi_*(G) \subset \pi_1(M)$ and let $\bar{\Gamma} = \tilde{\Gamma}/\alpha^{-1}(G)$, where $\tilde{\Gamma} \rightarrow \Gamma$ is the universal

cover. Remember that a map $f: Y \rightarrow P$ lifts to a map $\bar{f}: Y \rightarrow \bar{P}$ if and only if $f_*(\pi_1(Y)) \subset \pi_1(\bar{P}) = G$ and likewise for maps to M , respectively \bar{M} . In particular the projection π lifts to a map $\bar{\pi}: \bar{P} \rightarrow \bar{M}$, since $\pi_*(\pi_1(\bar{P})) = \pi_*(G) = \pi_1(\bar{M})$.

Lemma 2.33. $\bar{\pi}: \bar{P} \rightarrow \bar{M}$ is a principal $\bar{\Gamma}$ -bundle.

Proof. Denote the basepoint in \bar{P} by \bar{p}_0 and the induced basepoint in P by p_0 . Let γ be a loop in P with $\gamma(0) = p_0$, η a loop in Γ starting at the identity. Since $\gamma \cdot \eta$ is homotopic to the concatenation of γ with $p_0 \cdot \eta$, the action map $P \times \Gamma \rightarrow P$ induces

$$\pi_1(P) \times \pi_1(\Gamma) \rightarrow \pi_1(P) \quad ; \quad (a, b) \mapsto a \alpha(b)$$

on fundamental groups. Since $\pi_1(\bar{P}) \alpha(\pi_1(\bar{\Gamma})) = G \alpha(\alpha^{-1}(G)) = G = \pi_1(\bar{P})$, $P \times \Gamma \rightarrow P$ lifts to $\bar{P} \times \bar{\Gamma} \rightarrow \bar{P}$. This lift of the action is defined as follows: Choose two paths $\bar{\gamma}$ from \bar{p}_0 to \bar{p} in \bar{P} and $\bar{\eta}$ from 1 to \bar{g} in $\bar{\Gamma}$. $\bar{\gamma}$ and $\bar{\eta}$ project to $\gamma: I \rightarrow P$ and $\eta: I \rightarrow \Gamma$ respectively. $\gamma \cdot \eta$ has a lift $\bar{\gamma} \cdot \bar{\eta}$ to \bar{p}_0 . Then

$$(12) \quad \bar{p} \cdot \bar{g} = \bar{\gamma} \cdot \bar{\eta}(1) .$$

The fact that this is a well-defined continuous map follows from basic covering theory and it is easily seen to satisfy $\bar{p} \cdot (\bar{g}_1 \cdot \bar{g}_2) = (\bar{p} \cdot \bar{g}_1) \cdot \bar{g}_2$ and $\bar{p} \cdot 1 = \bar{p}$.

Consider the pullback of $\bar{M} \rightarrow M$ to P , i.e. $P \times_M \bar{M} \rightarrow P$. This cover is classified by the subgroup

$$\pi_*^{-1}(\pi_* G) = \alpha(\pi_1(\Gamma))G \subset \pi_1(P)$$

and is itself covered by $\bar{P} \rightarrow P \times_M \bar{M}$, which maps the point $\bar{p} \in \bar{P}$ over $p \in P$ to $(p, \bar{\pi}(\bar{p}))$. Since G is normal in $\alpha(\pi_1(\Gamma))G$ the deck transformation group A of the latter cover can be identified as a subquotient of $\pi_1(P)$ with:

$$A = \alpha(\pi_1(\Gamma))G/G = \alpha(\pi_1(\Gamma))/\alpha(\pi_1(\Gamma)) \cap G = \alpha(\pi_1(\Gamma))/\alpha^{-1}(G) .$$

If $\bar{g} \in \pi_1(\Gamma)/\alpha^{-1}(G) \subset \bar{\Gamma}$ the action (12) maps the fiber over $(p, \bar{m}) \in P \times_M \bar{M}$ into itself and sends \bar{p}_0 to $\bar{p}_0 \alpha(\bar{g})$, therefore it coincides with the (left) action of the deck transformation group. The projection $\bar{\pi}: \bar{P} \rightarrow \bar{M}$ factors as $\bar{P} \rightarrow P \times_M \bar{M} \rightarrow \bar{M}$. $\bar{P} \rightarrow P \times_M \bar{M}$ is a principal $\pi_1(\Gamma)/\alpha^{-1}(G)$ -bundle, $P \times_M \bar{M} \rightarrow \bar{M}$ is a principal Γ -bundle and the action of $\bar{\Gamma}$ satisfies: $\bar{\pi}(\bar{p} \cdot \bar{g}) = p \cdot g$, where \bar{p}, \bar{g} lie in the fiber above p and g . Therefore we are in the situation of lemma 2.32 for the central extension

$$1 \rightarrow \pi_1(\Gamma)/\alpha^{-1}(G) \rightarrow \bar{\Gamma} \rightarrow \Gamma \rightarrow 1 . \quad \square$$

Example 2.34. If $G = \{1\}$, then the previous lemma reveals the universal cover \tilde{P} to be a principal $\bar{\Gamma} = \tilde{\Gamma}/\ker \alpha$ -bundle over \tilde{M} . Of course, we regain the principal Γ -bundle $P \rightarrow M$ if we choose $G = \pi_1(P)$.

The situation in the proof can be turned upside down to determine whether lifting a principal Γ -bundle to a principal $\bar{\Gamma}$ -bundle is possible over some cover \bar{M} of M . Let $\pi: P \rightarrow M$ be a principal Γ -bundle and let $H \subset \pi_1(M)$ be a subgroup classifying a cover $\bar{\varrho}: \bar{M} \rightarrow M$. Denote by $\alpha: \pi_1(\Gamma) \rightarrow \pi_1(P)$ the map induced by the fiber inclusion and let $A \supset \ker \alpha$ be a subgroup of $\pi_1(\Gamma)$ classifying the cover $\bar{\Gamma} \rightarrow \Gamma$, which is again a Lie group. In the following corollary we consider extensions of H by $A/\ker \alpha$ of the form

$$(13) \quad 1 \longrightarrow A/\ker \alpha \xrightarrow{\alpha} G \xrightarrow{\pi_*} H \longrightarrow 1$$

for some subgroup $G \in \pi_1(P)$. Note that the maps in this extension are fixed to be the restrictions of $\pi_*: \pi_1(P) \rightarrow \pi_1(M)$ and $\alpha: \pi_1(\Gamma) \rightarrow \pi_1(P)$ to G and A respectively.

Definition 2.35. Let A , G and H be subgroups like above fitting into the short exact sequence (13), then the triple (A, G, H) will be called an extension of H by A in $\pi_1(P)$.

Corollary 2.36. Let $A = \pi_1(\bar{\Gamma})$ and $H = \pi_1(\bar{M})$, then $\bar{\varrho}^*P$ lifts to a principal $\bar{\Gamma}$ -bundle if and only if there is an extension (A, G, H) in $\pi_1(P)$. Moreover, extensions (A, G, H) in $\pi_1(P)$ are in 1 : 1-correspondence with the (pointed) lifts of $\bar{\varrho}^*P$.

Proof. Suppose (A, G, H) is an extension of H by A in $\pi_1(P)$. Since α maps A to G , the inclusion $A \subset \pi_1(\Gamma)$ factors as $A \subset \alpha^{-1}(G) \subset \pi_1(\Gamma)$. But $A/\ker \alpha$ and $\alpha^{-1}(G)/\ker \alpha$ coincide when injected into G via α , since they coincide with $\ker(\pi_*|_G)$. Therefore an application of the five lemma yields $A = \alpha^{-1}(G)$. By lemma 2.33 the subgroup $G \subset \pi_1(P)$ yields a principal $\bar{\Gamma}$ -bundle $\bar{P} \rightarrow \bar{M}$, which lifts $\bar{\varrho}^*P$.

Let $\bar{P} \rightarrow \bar{\varrho}^*P$ be a lift. $\bar{\varrho}^*P = P \times_M \bar{M} \rightarrow P$ is a covering map and since all spaces are connected and semi-locally simply-connected, the composition $\bar{P} \rightarrow P$ is a cover as well, which is classified by a subgroup $G \subset \pi_1(P)$. By the homotopy exact sequence of the corresponding fibrations

$$\begin{array}{ccccc} \bar{\Gamma} & \longrightarrow & \bar{P} & \longrightarrow & \bar{M} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma & \longrightarrow & P & \longrightarrow & M \end{array}$$

the triple (A, G, H) with $A = \pi_1(\bar{\Gamma})$, $H = \pi_1(\bar{M})$ provides an extension in $\pi_1(P)$. These constructions are inverse to each other. \square

Example 2.37. If $A = \ker \alpha$ and $H = \pi_1(\bar{M}) \subset \pi_1(M)$, then an extension in $\pi_1(P)$ corresponds to a subgroup $G \subset \pi_1(P)$, such that $\pi_*(G) = \pi_1(\bar{M})$. Phrased differently: In this case an extension in $\pi_1(P)$ is induced by a splitting $\pi_1(\bar{M}) \xrightarrow{\sim} G \subset \pi_1(P)$ and these splittings are in 1 : 1-correspondence with lifts of the principal Γ -bundle $\pi: \bar{\varrho}^*P \rightarrow \bar{M}$ to principal $\bar{\Gamma}$ -bundles $\bar{P} \rightarrow \bar{M}$.

Example 2.38. If $H = \{1\} \subset \pi_1(M)$, then there is a unique extension (A, G, H) in $\pi_1(P)$ given by $G = \alpha(A)$. Therefore $\bar{\varrho}^*P \rightarrow \bar{M}$ has a unique lift to a principal $\bar{\Gamma}$ -bundle. The case most interesting to us is where $P = P_{SO}$ is the frame bundle of M . Suppose for simplicity that $\dim(M) \geq 3$. Then we gain that \bar{M} carries a unique spin structure if and only if $\alpha: \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(P_{SO})$ has no kernel (compare with [31, lemma 2.10]).

3. TWISTED HILBERT A -MODULE BUNDLES

With only minor modifications we can extend the formalism of bundle gerbe modules to bundles with fibers not only vector spaces, but Hilbert C^* -modules for a C^* -algebra A (see [18] for an introduction to Hilbert C^* -modules).

Definition 3.1. Let M be a smooth manifold, A be a C^* -algebra and let $Y \rightarrow M$ be a surjective submersion. Let $L \rightarrow Y^{[2]}$ be a bundle gerbe. A (right) Hilbert

A -module bundle $E \rightarrow Y$ together with an action

$$\gamma: L \otimes \pi_2^* E \rightarrow \pi_1^* E$$

is called a *twisted Hilbert A -module bundle* for L if the following associativity diagram commutes:

$$\begin{array}{ccc} (\pi_{12}^* L \otimes \pi_{23}^* L) \otimes \pi_3^* E & \xlongequal{\quad} & \pi_{12}^* L \otimes (\pi_{23}^* L \otimes \pi_3^* E) \\ \downarrow \mu \otimes \text{id} & & \downarrow \text{id} \otimes \gamma \\ \pi_{13}^* L \otimes \pi_3^* E & & \pi_{12}^* L \otimes \pi_2^* E \\ & \searrow \gamma & \swarrow \gamma \\ & \pi_1^* E & \end{array}$$

E will be called *finitely generated* and *projective*, if its fibers are finitely generated and projective as Hilbert A -modules.

A morphism $\varphi: E \rightarrow E'$ of twisted Hilbert A -module bundles has to intertwine the two twistings as in definition 2.19.

Definition 3.2. Let E, E' be two bundle gerbe modules with respect to the same bundle gerbe L and denote the twistings by γ and γ' . A right A -linear map $f: E \rightarrow E'$ will be called a *morphism of twisted Hilbert A -module bundles* or (*twisted morphism* for short) if the following diagram commutes:

$$\begin{array}{ccc} L \otimes \pi_2^* E & \xrightarrow{\gamma} & \pi_1^* E \\ \downarrow \text{id}_L \otimes \pi_2^* f & & \downarrow \pi_1^* f \\ L \otimes \pi_2^* E' & \xrightarrow{\gamma'} & \pi_1^* E' \end{array}$$

Remark 3.3. Just as for ordinary bundle gerbe modules the homomorphism bundle $\text{Hom}(E, E')$ descends to a bundle $\text{hom}(E, E')$ over M . In particular, the bundle $\text{end}(E)$ is a bundle of C^* -algebras over M and its continuous sections $C(M, \text{end}(E))$ equipped with the sup-norm form a C^* -algebra. Moreover, the direct sum of two twisted Hilbert A -module bundles $E \rightarrow Y$ and $E' \rightarrow Y$ is defined to be the ordinary direct sum $E \oplus E' \rightarrow Y$ with the diagonal twisting $\gamma \oplus \gamma'$.

Example 3.4. Let A be a unital C^* -algebra and denote by $U(A)$ the group of unitary elements in A equipped with the norm topology (which due to unitality agrees with the strict topology on A). Like in example 2.17, let $P \rightarrow M$ be a principal Γ -bundle and let L be the lifting bundle gerbe associated to a central S^1 -extension

$$1 \rightarrow S^1 \rightarrow \widehat{\Gamma} \rightarrow \Gamma \rightarrow 1.$$

Suppose that $\widehat{\tau}: \widehat{\Gamma} \rightarrow U(A)$ satisfies $\widehat{\tau}(z\widehat{g}) = z\widehat{\tau}(\widehat{g})$ for all $\widehat{g} \in \widehat{\Gamma}$ and $z \in S^1$. Let $n \in \mathbb{N}$ and denote the induced diagonal representation in $U(M_n(A))$ by $\widehat{\tau}_n: \widehat{\Gamma} \rightarrow U(M_n(A))$. Similar to example 2.17 the bundle

$$\underline{A}^n = P \times A^n$$

is a twisted Hilbert A -module bundle with the action of L given by

$$\gamma([\widehat{g}, \lambda] \otimes v) = \lambda \widehat{\tau}_n(\widehat{g}) v .$$

Let $PU(A) = U(A)/S^1$, correspondingly $PU(M_n(A)) = U(M_n(A))/S^1$. Again, let τ_n be the induced diagonal homomorphism $\Gamma \rightarrow PU(M_n(A))$ and set $\tau = \tau_1$. Via the adjoint action Ad_τ of $PU(A)$ on A we get an associated bundle of C^* -algebras $\mathcal{A} = P \times_{\text{Ad}_\tau} A$. More generally, we get $M_n(\mathcal{A}) = P \times_{\text{Ad}_{\tau_n}} M_n(A)$. The norm continuous sections $C(M, M_n(\mathcal{A})) \cong C(M, \mathcal{A}) \otimes M_n(\mathbb{C})$ form another C^* -algebra and we note that

$$\text{end}(\underline{A}^n) = M_n(\mathcal{A}) .$$

Example 3.5. The following construction is related to example 3.4. Let A , Γ , $\widehat{\tau}_n$ and τ_n be as above. Suppose $P \rightarrow M$ is a trivializable principal Γ -bundle, which implies that there exists an equivariant map $\sigma: P \rightarrow \Gamma$. Let V be a finitely generated, projective right Hilbert A -module, i.e. $V = tA^n$ for some projection $t \in M_n(A)$ and set

$$V_\sigma = \{(p, v) \in \underline{A}^n \mid \text{Ad}_{(\tau_n \circ \sigma)(p)^{-1}}(t)v = v\} .$$

To see that V_σ is locally trivial as a bundle of Hilbert A -modules over P at a point $p \in P$, choose an open neighborhood $U \subset \Gamma$ of $\sigma(p)$, such that there is a section $\kappa: U \rightarrow \widehat{\Gamma}$ of the projection $\widehat{\Gamma} \rightarrow \Gamma$. Choose an open neighborhood $U' \subset \sigma^{-1}(U)$ containing p and consider

$$V_\sigma|_{U'} \rightarrow U' \times V \quad ; \quad (p, v) \mapsto (p, (\widehat{\tau}_n \circ \kappa \circ \sigma)(p)v) .$$

This provides a trivialization of V_σ over U' around $p \in P$. The twisting δ of V_σ is the restriction of the twisting of \underline{A}^n , that is

$$\delta: L \otimes \pi_2^* V_\sigma \rightarrow \pi_1^* V_\sigma \quad ; \quad \delta([\widehat{g}, \lambda] \otimes v) = \lambda \widehat{\tau}_n(\widehat{g}) v$$

Indeed, let $(p, v) \in V_\sigma$ and $[\widehat{g}, \lambda] \in L_{(pg^{-1}, p)}$, then we have

$$\begin{aligned} \text{Ad}_{(\tau_n \circ \sigma)(pg^{-1})^{-1}}(t) \lambda \widehat{\tau}_n(\widehat{g}) v &= \lambda \widehat{\tau}_n(\widehat{g}) \text{Ad}_{(\tau_n \circ \sigma)(p)^{-1}}(t) \widehat{\tau}_n(\widehat{g}^{-1}) \widehat{\tau}_n(\widehat{g}) v \\ &= \lambda \widehat{\tau}_n(\widehat{g}) \text{Ad}_{(\tau_n \circ \sigma)(p)^{-1}}(t) v = \lambda \widehat{\tau}_n(\widehat{g}) v . \end{aligned}$$

Thus, $\delta([\widehat{g}, \lambda] \otimes v) = (pg^{-1}, \lambda \widehat{\tau}_n(\widehat{g}) v) \in V_\sigma$ and δ is well-defined. Note that V_σ embeds into \underline{A}^n via a twisted morphism.

Example 3.6. Let $L_i \rightarrow Y_i^{[2]}$ for $i \in \{1, 2\}$ be two bundle gerbes and $E \rightarrow Y_1$ be a twisted Hilbert A -module bundle for L_1 . Let $F \rightarrow Y_2$ be a bundle gerbe module for L_2 , then we can form the exterior tensor product

$$E \boxtimes F = \pi_1^* E \otimes \pi_2^* F ,$$

where $\pi_i: Y_1 \times_M Y_2 \rightarrow Y_i$ denotes the canonical projection. This is a twisted Hilbert A -module bundle for $L_1 \boxtimes L_2$.

The next lemma is the first step to reveal the connection between twisted Hilbert A -module bundles and the K -group $K_0(C(M, \mathcal{A}))$: Projective, finitely generated twisted Hilbert A -module bundles have stable inverses. This lemma is a straightforward generalization of the corresponding result for ordinary Hilbert A -module bundles.

Lemma 3.7. *Let $E \rightarrow P$ be a finitely generated, projective twisted Hilbert A -module bundle for a lifting bundle gerbe like in example 3.4, then there exists a natural number $n \in \mathbb{N}$ and a twisted Hilbert A -module bundle $F \rightarrow P$ such that*

$$E \oplus F \cong \underline{A}^n$$

via a morphism of twisted bundles.

Proof. Choose a finite open cover $\bigcup_{i \in I} U_i \supset M$ such that there exist local sections $\sigma_i: U_i \rightarrow P$ for the projection map $\pi: P \rightarrow M$. Let $E_i = \sigma_i^* E$ and suppose the U_i have been chosen small enough such that E_i is trivializable. Since E_i is a finitely generated projective Hilbert A -module bundle, there is a fiberwise A -linear isometric embedding

$$\varphi_i: E_i \rightarrow \underline{A}^N$$

for each $i \in I$, which factors through the trivialization $\bar{\varphi}_i: E_i \rightarrow \underline{V}$, where V is the typical fiber of E . Let $P|_{U_i} = \pi^{-1}(U_i)$. The sections σ_i yield diagonal embeddings

$$\Delta_i: P|_{U_i} \rightarrow P|_{U_i}^{[2]} \quad ; \quad p \mapsto (p, \sigma_i(\pi(p))) .$$

Likewise let $E|_{U_i} = \rho^{-1}(P|_{U_i})$, where $\rho: E \rightarrow P$ denotes the projection map. Let $\gamma: L \otimes \pi_2^* E \rightarrow \pi_1^* E$ be the twisting of E and $\delta: L \otimes \pi_2^* \underline{A}^N \rightarrow \pi_1^* \underline{A}^N$ be the one of \underline{A}^N . Restricting γ and δ to the image of Δ_i we get an A -linear fiberwise isometry

$$\psi_i: E|_{U_i} \xrightarrow{\gamma^{-1}} \Delta_i^* L \otimes \pi^* E_i \xrightarrow{\text{id} \otimes \pi^* \varphi_i} \Delta_i^* L \otimes \underline{A}^N \xrightarrow{\delta} \underline{A}^N$$

We have to check that each ψ_i is indeed a morphism of twisted Hilbert A -module bundles. Note that

$$\begin{aligned} (\Delta_i \circ \pi_2)(p, pg) &= (pg, \sigma_i(\pi(p))) , \\ (\Delta_i \circ \pi_1)(p, pg) &= (p, \sigma_i(\pi(p))) \end{aligned}$$

for every $g \in \Gamma$. Therefore the bundle gerbe multiplication $\mu: \pi_{12}^* L \otimes \pi_{23}^* L \rightarrow \pi_{13}^* L$ induces an isomorphism of line bundles

$$\mu_i: L \otimes \pi_2^* \Delta_i^* L \rightarrow \pi_1^* \Delta_i^* L .$$

By the associativity of the action of L on E and on \underline{A}^N respectively, the top and bottom square of the following diagram commute.

$$\begin{array}{ccc} L \otimes \pi_2^* E|_{U_i} & \xrightarrow{\gamma} & \pi_1^* E|_{U_i} \\ \downarrow \text{id}_L \otimes \pi_2^* \gamma^{-1} & & \downarrow \pi_1^* \gamma^{-1} \\ L \otimes \pi_2^* (\Delta_i^* L \otimes \pi^* E_i) & \xrightarrow{\mu_i \otimes \text{id}} & \pi_1^* (\Delta_i^* L \otimes \pi^* E_i) \\ \downarrow \text{id}_L \otimes \text{id} \otimes \pi_2^* \pi^* \varphi_i & & \downarrow \text{id} \otimes \pi_1^* \pi^* \varphi_i \\ L \otimes \pi_2^* (\Delta_i^* L \otimes \underline{A}^N) & \xrightarrow{\mu_i \otimes \text{id}} & \pi_1^* (\Delta_i^* L \otimes \underline{A}^N) \\ \downarrow \text{id}_L \otimes \pi_2^* \delta & & \downarrow \pi_1^* \delta \\ L \otimes \pi_2^* \underline{A}^N & \xrightarrow{\delta} & \pi_1^* \underline{A}^N \end{array}$$

Since the composition of the vertical maps on the left and on the right hand side correspond to $\text{id}_L \otimes \pi_2^* \psi_i$ and $\pi_1^* \psi_i$ respectively, this proves that ψ_i is indeed a twisted morphism. We can now use a partition of unity $h_i: M \rightarrow [0, 1]$ subordinate to the cover to patch the ψ_i together to form a twisted morphism: Let $k = |I|$, $n = N \cdot k$ and

$$(14) \quad \Psi: E \rightarrow \underline{A}^n \quad ; \quad v \mapsto \sum_{i \in I} h_i(\pi \circ \rho(v)) \cdot \psi_i(v) ,$$

where ψ_i is understood as an embedding into the i th summand of $\underline{A}^n = \bigoplus_I \underline{A}^N$. Just as in the case of ordinary vector bundles we take the orthogonal complement, i.e.

$$F = \{(v, p) \in \underline{A}^n \mid \langle v, \Psi(w) \rangle = 0 \ \forall w \in E_p\} ,$$

which comes equipped with a canonical projection map $\tilde{\rho}: F \rightarrow P$. Let $\langle \cdot, \cdot \rangle_p$ be the A -valued inner product in the fiber above p . Since twistings are isometric and Ψ is a twisted morphism, we have for $v \in F_p, w \in E_p, \hat{g} \in \hat{\Gamma}$ and $\lambda, \mu \in \mathbb{C}$

$$\begin{aligned} \langle \delta^k([\hat{g}, \lambda] \otimes v), \Psi(\gamma([\hat{g}, \mu] \otimes w)) \rangle_{pg} &= \langle \delta^k([\hat{g}, \lambda] \otimes v, \delta^k([\hat{g}, \mu] \otimes \Psi(w)) \rangle_{pg} \\ &= \mu \bar{\lambda} \langle v, \Psi(w) \rangle_p = 0 . \end{aligned}$$

From this it follows that δ^k restricts to $\gamma_F: L \otimes \pi_2^* F \rightarrow \pi_1^* F$.

Fix $p_0 \in P$. It remains to show that $\Psi(E_{p_0}) \oplus F_{p_0} = \underline{A}^n$, which cannot be taken for granted in the case of Hilbert A -modules. Furthermore, we have to prove that F is a locally trivial bundle over P . Since both problems are local, we can solve them using the same arguments as in the non-twisted case (see [30]). Let $J \subset I$ be the indices of those U_j with $p_0 \in \pi^{-1}(U_j)$ and let $l = |J|$. We define

$$U = \bigcap_{j \in J} \pi^{-1}(U_j) \subset P .$$

Let $g_j: P|_{U_j} \rightarrow \Gamma$ be the Γ -equivariant map such that $\sigma_j(\pi(p)) = p g_j(p)$ and let V be the typical fiber of E as above. We can apply the construction of example 3.5 to get twisted Hilbert A -module bundles $V_j = V_{g_j}$ over U . Let $W = V_1 \oplus \dots \oplus V_l$. Since the trivializations $\bar{\varphi}_j$ map $E_{\sigma_j(\pi(p))}$ to V , Ψ factors over a twisted morphism

$$\bar{\Psi}: E|_U \rightarrow W$$

Let W^\perp be the orthogonal complement of W in $\underline{A}^n = (\underline{A}^N)^k$. W^\perp is a twisted Hilbert A -module bundle, since $W^\perp = V_1^\perp \oplus \dots \oplus V_l^\perp \oplus (\underline{A}^N)^{k-l}$ and V_i^\perp arises from the construction in example 3.5 by exchanging the projection t by $1 - t$. From this, we also see that $W \oplus W^\perp = \underline{A}^n$. Let $\rho_W: W \rightarrow U$ be the projection and

$$\bar{F} = \{w \in W \mid \langle w, \bar{\Psi}(x) \rangle = 0 \ \forall x \in E_{\rho_W(w)}\} ,$$

then

$$F = \bar{F} \oplus W^\perp .$$

Since directness of the sum easily follows from positive definiteness of the inner product, it remains to show that $\bar{\Psi}(E_{p_0}) + \bar{F}_{p_0} = W_{p_0}$. Following the definition of $\bar{\Psi}$ we have

$$\bar{\Psi}(E_{p_0}) = \{(\lambda_1 \bar{\psi}_1(v), \dots, \lambda_l \bar{\psi}_l(v)) \in W \mid v \in E_{p_0}\}$$

for some $\lambda_i \in [0, 1]$. At least one of the λ_i does not vanish. Without loss of generality we will assume that $\lambda_1 \neq 0$. Since $\bar{\psi}_j: E|_U \rightarrow V_j|_U$ is a twisted isomorphism, this

can be rewritten as

$$\bar{\Psi}(E_{p_0}) = \{(x, \kappa_2(x), \dots, \kappa_l(x)) \in W \mid x \in (V_1)_{p_0}\} ,$$

where $\kappa_j: V_1|_U \rightarrow V_j|_U$ is defined by $\kappa_j = \lambda_1^{-1} \lambda_j \bar{\psi}_j \circ \bar{\psi}_1^{-1}$. Using the fiberwise adjoint of κ_j , \bar{F} turns out to be the A -linear span of $\bigcup_{j \in J} B_j$ with

$$B_j = \left\{ (-\kappa_j^*(v), 0, \dots, 0, \underbrace{v}_j, 0, \dots, 0) \in W \mid v \in V_j|_U \right\} .$$

To show that $\bar{\Psi}(E_{p_0}) + \bar{F}_{p_0} = W_{p_0}$ we have to solve the following system of equations

$$\begin{aligned} v_1 - \kappa_2^*(v_2) - \dots - \kappa_l^*(v_l) &= w_1 \\ v_2 + \kappa_2(v_1) &= w_2 \\ &\vdots \\ v_l + \kappa_l(v_1) &= w_l \end{aligned}$$

for $v_i \in (V_i)_{p_0}$, which is equivalent to

$$\begin{aligned} v_1 + \kappa_2^*(\kappa_2(v_1)) + \dots + \kappa_l^*(\kappa_l(v_1)) &= w_1 + \kappa_2^*(w_2) + \dots + \kappa_l^*(w_l) \\ v_2 &= w_2 - \kappa_2(v_1) \\ &\vdots \\ v_l &= w_l - \kappa_l(v_1) \end{aligned}$$

and therefore can be solved since $1 + \kappa_2^* \circ \kappa_2 + \dots + \kappa_l^* \circ \kappa_l$ is a strictly positive element in $\text{End}((V_1)_{p_0})$. Concerning the local triviality of \bar{F} , the map

$$\bar{F}|_U \rightarrow V_2 \oplus \dots \oplus V_l \quad ; \quad (v_1, v_2, \dots, v_l) \mapsto (v_2, \dots, v_l)$$

provides a twisted isomorphism from $\bar{F}|_U$ to $V_2 \oplus \dots \oplus V_l$, because \bar{F} is spanned by the B_j . Since the latter is locally trivial, \bar{F} is as well. This finishes the proof. \square

Definition 3.8. Let A , Γ and \mathcal{A} be as in example 3.4. The category of twisted Hilbert A -module bundles over M and twisted morphisms will be denoted by $\text{TwBun}_{\mathcal{A}}(M)$. A *virtual twisted Hilbert A -module bundle* is a class of pairs (E_+, E_-) with $E_+, E_- \in \text{TwBun}_{\mathcal{A}}(M)$ denoted by $E_+ - E_-$ subject to the following equivalence relation: $(E_+, E_-) \sim (F_+, F_-)$ iff $\exists G \in \text{TwBun}_{\mathcal{A}}(M)$ such that $E_+ \oplus F_- \oplus G \cong E_- \oplus F_+ \oplus G$. $K_{\mathcal{A}}^0(M)$ is the group of isomorphism classes of virtual twisted Hilbert A -module bundles. Lemma 3.7 shows that every element in $K_{\mathcal{A}}^0(M)$ can be represented by a pair of the form (E, \underline{A}^N) , in which the second entry is trivial.

Theorem 3.9. *The category $\text{TwBun}_{\mathcal{A}}(M)$ is naturally equivalent to the category of finitely generated projective $C(M, \mathcal{A})$ -modules. In particular*

$$K_{\mathcal{A}}^0(M) \cong K_0(C(M, \mathcal{A})) .$$

Proof. Let $t \in M_n(C(M, \mathcal{A})) = C(M, M_n(\mathcal{A}))$ be a projection-valued section of $M_n(\mathcal{A})$. Since $M_n(\mathcal{A})$ corresponds to $P \times_{\tau_n} M_n(A)$, t can be identified with a Γ -equivariant map $t: P \rightarrow M_n(A)$, i.e. $t(pg) = \text{Ad}_{\tau_n(g)^{-1}}(t(p))$. Let

$$E = \{(p, v) \in P \times A^n \mid t(p)v = v\} .$$

Choose $p_0 \in P$, let $V = t(p_0)A^n$ and consider $\Phi: E \rightarrow P \times V$; $(p, v) \mapsto (p, t(p_0)v)$ and $\Psi: P \times V \rightarrow E$; $(p, v) \mapsto (p, t(p)v)$. At p_0 we have $\Psi \circ \Phi = \text{id}_E$ and $\Phi \circ \Psi =$

$\text{id}_{P \times V}$. So, if we think of $\Psi \circ \Phi$ as an element in $C(P, \text{End}(E))$, it is invertible in a neighborhood $U \subset P$ containing p_0 , since invertible elements form an open subset in a C^* -algebra. Likewise $\Phi \circ \Psi$ is invertible in some open neighborhood $U' \subset P$. Thus, Φ and Ψ are isomorphisms over some open subset $\tilde{U} \subset P$ containing p_0 , which implies that E is locally trivial. Moreover, it carries a twisting of the form

$$\gamma: L \otimes \pi_2^* E \rightarrow \pi_1^* E \quad ; \quad \gamma([\hat{g}, \lambda] \otimes v) = \lambda \hat{\tau}_n(\hat{g}) v ,$$

which is well-defined, since for $\hat{g} \in \hat{\Gamma}$ and $v \in E_p$ we have

$$t(pg^{-1}) \hat{\tau}_n(\hat{g}) v = \hat{\tau}_n(\hat{g}) t(p) \hat{\tau}_n(\hat{g}^{-1}) \hat{\tau}_n(\hat{g}) v = \hat{\tau}_n(\hat{g}) t(p) v = \hat{\tau}_n(\hat{g}) v .$$

If on the other hand E is a twisted Hilbert A -module bundle for L over P , then by lemma 3.7 it can be embedded via a twisted morphism into \underline{A}^n as a direct summand, i.e. $E \rightarrow E \oplus F \cong \underline{A}^n$. Let $t \in C(P, \text{End}(A^n))$ be the projection to E for each point $p \in P$. Since t corresponds to a twisted morphism it descends to a global projection valued section of $\text{end}(\underline{A}^n) = C(M, M_n(\mathcal{A}))$.

It is easy to check that these two constructions are inverse to each other, which proves the second part of the theorem, i.e. $K_{\mathcal{A}}^0(M) \cong K_0(C(M, \mathcal{A}))$. Let $\text{FinProj}(B)$ be the category of finitely generated projective Hilbert B -modules for a C^* -algebra B . To see the natural equivalence

$$\text{FinProj}(C(M, \mathcal{A})) \cong \text{TwBun}_{\mathcal{A}}(M) ,$$

fix for every twisted Hilbert A -module bundle E an embedding $\phi_E: E \rightarrow E \oplus F_E \cong \underline{A}^{n_E}$ as a direct summand into \underline{A}^{n_E} (this involves the axiom of global choice for proper classes). Let $t_E \in C(M, M_{n_E}(\mathcal{A}))$ be the projection valued section from the construction above. The functor $\text{TwBun}_{\mathcal{A}}(M) \rightarrow \text{FinProj}(C(M, \mathcal{A}))$ sends an object E to the Hilbert $C(M, \mathcal{A})$ -module $\mathcal{V}_E = t_E \cdot C(M, \mathcal{A}^{n_E})$. Let E' be another twisted Hilbert A -module bundle. A twisted morphism corresponds to a section of $\text{hom}(E, E')$, which embeds into the upper left corner of $\text{hom}(E \oplus F_E, E' \oplus F_{E'})$. In fact,

$$\begin{aligned} C(M, \text{hom}(E, E')) &\cong t_{E'} C(M, \text{hom}(\underline{A}^{n_E}, \underline{A}^{n_{E'}})) t_E \\ &\cong t_{E'} C(M, M_{n_{E'} \times n_E}(\mathcal{A})) t_E \\ &\cong \text{Hom}(\mathcal{V}_E, \mathcal{V}_{E'}) , \end{aligned}$$

where the last line denotes the $C(M, \mathcal{A})$ -linear adjointable operators. This isomorphism yields the value of the functor on morphisms and our considerations above show that it is an equivalence of categories. \square

Corollary 3.10. *If L is the lifting bundle gerbe of a principal $PU(n)$ -bundle and \mathcal{K} the associated matrix bundle, then we get isomorphisms*

$$K_L^0(M) = K_{\mathcal{K}}^0(M) \cong K_0(C(M, \mathcal{K})) .$$

Remark 3.11. A slight drawback of the above construction is that the functor depends up to natural isomorphism on the choice of embedding $E \rightarrow \underline{A}^n$, whereas in the non-twisted case a Hilbert A -module bundle E over M can be sent to the global sections $C(M, E)$. In case of trivial twisting (i.e. $dd(L) = 0$) the commutative C^* -algebra $C(M)$, that is the sections in the trivial bundle $M \times \mathbb{C} \rightarrow M$ of C^* -algebras provides a canonical realization of the twist, whereas for $dd(L) \neq 0$ there is no such choice of a unital C^* -algebra, but only up to Morita equivalence.

3.1. Countertwisting and twisted K -homology. Let \mathcal{A} be a bundle of C^* -algebras with typical fiber A associated to a principal Γ -bundle P like in example 3.4 and let $dd(L) \in H^3(M, \mathbb{Z})$ be a torsion element. It follows from proposition 2.1 (v) in [2] that in this case there is a bundle of matrix algebras \mathcal{K} over M associated to a principal $PU(N)$ -bundle $P_{\mathcal{K}}$ such that its lifting bundle gerbe $L_{\mathcal{K}}$ satisfies $dd(L_{\mathcal{K}}) = dd(L)$. By a choice of a trivialization Q of $L^* \boxtimes L_{\mathcal{K}}$ it is possible to shift the twisting of the algebra \mathcal{A} to the matrix algebra \mathcal{K} in the following sense

Theorem 3.12. *Let \mathcal{A} , \mathcal{K} and Q be as above, then the following K -groups are isomorphic via a Morita equivalence:*

$$K_{\mathcal{A}}^0(M) \cong K_0(C(M, \mathcal{A})) \cong K_0(C(M, \mathcal{K}) \otimes A) .$$

In particular, the image of $[E] \in K_{\mathcal{A}}^0(M)$ in $K_0(C(M, \mathcal{K}) \otimes A)$ is given by the Fredholm module $[C(M, Q(S^ \boxtimes E))]$, where*

$$S = P_{\mathcal{K}} \times \mathbb{C}^N$$

is the bundle gerbe module for $L_{\mathcal{K}}$ from example 2.17 using the regular representation of $U(N)$ on \mathbb{C}^N .

Proof. There is a Morita equivalence between A and $M_N(A) = M_N(\mathbb{C}) \otimes A$ induced by the canonical imprimitivity bimodule $(\mathbb{C}^N)^* \otimes A$, which carries a right $M_N(\mathbb{C}) \otimes A$ - and a left A -action and corresponding inner products. Therefore the fibers of $S^* \boxtimes \underline{A}$ over $P_{\mathcal{K}} \times_M P$ are Hilbert $M_N(\mathbb{C}) \otimes A$ -modules. By our hypothesis $dd(L^* \boxtimes L_{\mathcal{K}}) = 0$, thus by corollary 2.27 the bundle $S^* \boxtimes \underline{A}$ descends to $\mathcal{V} = Q(S^* \boxtimes \underline{A})$ over M . Since the right action of $M_N(\mathbb{C})$ gets twisted by the adjoint action of $U(N)$ in this process, the sections $C(M, \mathcal{V})$ form a right Hilbert $C(M, \mathcal{K}) \otimes A$ -module, where $C(M, \mathcal{K}) = C(M, \text{end}(S^*))$ acts only on the S^* -factor of \mathcal{V} . Likewise, the left action of A on $S^* \boxtimes \underline{A}$ gets twisted by the adjoint action Ad_{τ} after the descend and therefore turns $C(M, \mathcal{V})$ into a left Hilbert $C(M, \mathcal{A})$ -module. That this is an imprimitivity bimodule can be deduced from the local situation via a partition of unity. Thus, $C(M, \mathcal{V})$ induces an isomorphism on K -theory explicitly given by

$$\begin{aligned} K_0(C(M, \mathcal{A})) &\rightarrow K_0(C(M, \mathcal{K} \otimes A)) \\ [W] &\mapsto [W \otimes_{C(M, \mathcal{A})} C(M, \mathcal{V})] \end{aligned}$$

Let $t \in C(M, M_n(\mathcal{A}))$ be a projection, then, in particular, the homomorphism sends $t C(M, \mathcal{A}^n)$ to $t C(M, \mathcal{V}^n)$. Since every twisted Hilbert A -module bundle E is twistedly isomorphic to one of the form

$$t \underline{A}^n := \{(p, v) \in P \times A^n \mid t(p)v = v\}$$

(see the proof of lemma 3.7), $[E] \in K_{\mathcal{A}}^0(M)$ gets mapped to the equivalence class of

$$t C(M, \mathcal{V}^n) \cong C(M, Q(S^* \otimes t \underline{A}^n)) \cong C(M, Q(S^* \otimes E)) . \quad \square$$

Remark 3.13. Note that $L^* \boxtimes L_{\mathcal{K}}$ is the lifting bundle gerbe of the central S^1 -extension

$$1 \rightarrow S^1 \rightarrow U(N) \bar{\otimes} \widehat{\Gamma} \rightarrow PU(N) \times \Gamma \rightarrow 1$$

for the principal $PU(N) \times \Gamma$ -bundle $P_{\mathcal{K}} \times_M P$, where $U(N) \bar{\otimes} \widehat{\Gamma}$ denotes the quotient of $U(N) \times \widehat{\Gamma}$ by the diagonal action of S^1 , i.e. $(uz, \widehat{g}) \sim (u, z^{-1}\widehat{g})$ for $\widehat{g} \in \widehat{\Gamma}$, $u \in U(N)$ and $z \in S^1$. Since Q is a trivialization of $L^* \boxtimes L_{\mathcal{K}}$, there is an isomorphism

$$L^* \boxtimes L_{\mathcal{K}} \otimes \pi_2^* Q \rightarrow \pi_1^* Q .$$

This induces an action of $U(N) \bar{\otimes} \widehat{\Gamma}$ on the principal S^1 -bundle $\widehat{P} \rightarrow P_{\mathcal{K}} \times_M P$ of Q , which is compatible with the action of $PU(N) \times \Gamma$ on P and turns \widehat{P} into a principal $U(N) \bar{\otimes} \widehat{\Gamma}$ -bundle by lemma 2.32. Consider the representation κ of $U(N) \bar{\otimes} \widehat{\Gamma}$ on $(\mathbb{C}^N)^* \otimes A$ given by

$$[u, \widehat{g}] \cdot (\xi \otimes a) = (\xi \circ u^*) \otimes \widehat{\tau}(\widehat{g}) a .$$

The bundle \mathcal{V} coincides with the bundle of right Hilbert A -modules $\widehat{P} \times_{\kappa} (\mathbb{C}^N)^* \otimes A$, which could also be used to prove the claims of the last lemma.

Definition 3.14. A choice of the matrix bundle \mathcal{K} and the bundle gerbe module S as in the previous lemma will be called a *countertwisting* for \mathcal{A} .

3.1.1. *Chern character.* Let $L \rightarrow Y^{[2]}$ be a bundle gerbe with $dd(L) \in H^3(M, \mathbb{Z})$ a torsion class and equipped with a twisted connection ∇^L . Let $F \rightarrow Y$ be a bundle gerbe module for L with connection ∇^F . Let $f \in \Omega^2(Y)$ be a closed curving for ∇^L , which exists since $dd^{\mathbb{R}}(L) = 0$ (see remark 2.15). As stated in remark 2.23 we have $\Omega_F - f = \pi^* \omega_f$ for a 2-form $\omega_f \in \Omega^2(M, \text{end}(F))$. ω_f is closed, since f was chosen to be closed and π^* is injective. Every invariant polynomial P defines a closed form $P(\omega_f) \in \Omega^{2*}(M)$ [6, 33].

Definition 3.15. Let $\exp\left(\frac{i\omega_f}{2\pi}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i}{2\pi}\right)^k \underbrace{\omega_f \wedge \cdots \wedge \omega_f}_{k \text{ times}} \in \Omega^*(M, \text{end}(F))$,

then we define the *Chern character* $\text{ch}_f(F)$ to be

$$\text{ch}_f(F) = \text{tr} \left(\exp \left(\frac{i\omega_f}{2\pi} \right) \right) \in H^{2*}(M, \mathbb{R}) .$$

If L is a lifting bundle gerbe for a flat central extension and ∇^L the canonical flat connection, then f can be chosen to vanish. In this case we drop f from the notation.

If $L_1 \rightarrow Y_1^{[2]}$ and $L_2 \rightarrow Y_2^{[2]}$ are two bundle gerbes and E_i for $i \in \{1, 2\}$ is a bundle gerbe module for L_i , then $E_1 \boxtimes E_2 \rightarrow Y_1 \times_M Y_2$ is a module for $L_1 \boxtimes L_2$. This operation induces a multiplicative operation on twisted K -theory of the form

$$K_{L_1}^0(M) \times K_{L_2}^0(M) \rightarrow K_{L_1 \boxtimes L_2}^0(M) ,$$

which has already been observed in the foundational work of Donovan and Karoubi in [9]. If L_i is equipped with a connection ∇^{L_i} with curvature Ω_{L_i} , then the curvature $\Omega_{L_1 \boxtimes L_2}$ of the canonical connection $\nabla^{L_1 \boxtimes L_2}$ is the sum of the two curvature forms. Thus, if f_i is a curving for ∇^{L_i} , then $f_1 + f_2$ is a curving for $\nabla^{L_1 \boxtimes L_2}$.

Theorem 3.16. *The Chern character ch_f yields a group homomorphism $K_L^0(M) \rightarrow H^{2*}(M, \mathbb{R})$ from bundle gerbe twisted K -theory to the even cohomology groups over the reals, which is multiplicative in the sense that*

$$\text{ch}_{f_1+f_2}(E_1 \boxtimes E_2) = \text{ch}_{f_1}(E_1) \cup \text{ch}_{f_2}(E_2)$$

for bundle gerbe modules E_i like in the previous paragraph.

Proof. The first statement is obvious, since connections are well-behaved with respect to direct sums. The second claim follows from the observation that

$$\exp \left(\frac{i(\omega_{f_1} \otimes \text{id}_{E_2} + \text{id}_{E_1} \otimes \omega_{f_2})}{2\pi} \right) = \exp \left(\frac{i\omega_{f_1}}{2\pi} \right) \wedge \exp \left(\frac{i\omega_{f_2}}{2\pi} \right) ,$$

where the wedge product of $\eta \in \Omega^p(M, \text{end}(E_1))$ and $\tau \in \Omega^q(M, \text{end}(E_2))$ is defined to be

$$(\eta \wedge \tau)(X_1, \dots, X_{p+q}) = \sum_{\sigma \in S(p,q)} \text{sign}(\sigma) \eta(X_{\sigma(1)}, \dots, X_{\sigma(p)}) \otimes \tau(X_{\sigma(p+1)}, \dots, X_{\sigma(p+q)}) .$$

Since the trace on the fibers of $\text{end}(E_1) \otimes \text{end}(E_2)$ is the product of the two traces, the right hand side evaluates to the wedge product of the forms representing the Chern characters $\text{ch}_{f_i}(E_i)$ for $i \in \{1, 2\}$. \square

In case L is the lifting bundle gerbe of a principal $PU(n)$ -bundle P and \mathcal{K} is the matrix bundle associated to P , then we get an induced homomorphism

$$\text{ch}: K_0(C(M, \mathcal{K})) \rightarrow H^{2*}(M, \mathbb{R}) .$$

This can be used to extend the definition to a Chern character on twisted Hilbert A -module bundles, which takes values in the cohomology ring $H^{2*}(M, K_0(A) \otimes \mathbb{R})$ via the Künneth theorem of operator algebraic K -theory. More precisely, let \mathcal{A} , P , \mathcal{K} and Q be as described at the beginning of section 3.1. By theorem 3.12 and the Künneth formula we have

$$\begin{aligned} K_{\mathcal{A}}^0(M) \otimes \mathbb{R} &\cong K_0(C(M, \mathcal{K}) \otimes A) \otimes \mathbb{R} \\ &\cong K_0(C(M, \mathcal{K})) \otimes K_0(A) \otimes \mathbb{R} \oplus K_1(C(M, \mathcal{K})) \otimes K_1(A) \otimes \mathbb{R} , \end{aligned}$$

where the first isomorphism is induced by the trivialization Q of $L^* \boxtimes L_{\mathcal{K}}$.

Definition 3.17. We define the Chern character of a twisted Hilbert A -module bundle $E \rightarrow P$ depending on a choice of a trivialization Q of $L^* \boxtimes L_{\mathcal{K}}$ as in section 3.1 to be

$$\text{ch}_Q: K_{\mathcal{A}}^0(M) \rightarrow K_0(C(M, \mathcal{K})) \otimes K_0(A) \otimes \mathbb{R} \rightarrow H^{2*}(M, K_0(A) \otimes \mathbb{R}) ,$$

where the first map is the projection to the first Künneth summand and the second is the Chern character described above applied to the first tensor factor.

Since the exterior tensor product of a twisted Hilbert A -module bundle with a bundle gerbe module is again a twisted Hilbert A -module bundle in a canonical way – a fact that was already exploited in section 3.1 – there is a homomorphism

$$(15) \quad K_{\mathcal{L}}^0(M) \times K_{\mathcal{A}}^0(M) \rightarrow K_{\mathcal{L} \otimes \mathcal{A}}^0(M) \quad , \quad (F, E) \mapsto F \boxtimes E$$

where \mathcal{L} now is an arbitrary bundle of *matrix* algebras over M . The Künneth homomorphism is compatible with this multiplication in the sense that the following diagram commutes

$$\begin{array}{ccc} K_{\mathcal{L}}^0(M) \times K_{\mathcal{A}}^0(M) & \xrightarrow{\quad \quad \quad} & K_{\mathcal{L} \otimes \mathcal{A}}^0(M) \\ \downarrow & & \downarrow \\ K_0(C(M, \mathcal{L})) \times K_0(C(M, \mathcal{K})) \otimes K_0(A) \otimes \mathbb{R} & \rightarrow & K_0(C(M, \mathcal{K} \boxtimes \mathcal{L})) \otimes K_0(A) \otimes \mathbb{R} \end{array}$$

Since the lifting bundle gerbe $L_{\mathcal{L}}$ also provides a trivialization of $L_{\mathcal{L}}^* \boxtimes L_{\mathcal{L}}$ which is flat, we immediately get the following corollary of theorem 3.16.

Corollary 3.18. *Let \mathcal{L} , \mathcal{A} , \mathcal{K} , $L_{\mathcal{L}}$ and Q be as above, then the Chern character $\text{ch}_Q: K_{\mathcal{A}}^0(M) \rightarrow H^{2*}(M, \mathbb{R})$ is multiplicative in the sense that*

$$\text{ch}_{L_{\mathcal{L}} \boxtimes Q}(F \boxtimes E) = \text{ch}(F) \cup \text{ch}_Q(E) .$$

If \mathcal{A} is a bundle of C^* -algebras such that the associated lifting bundle gerbe L has trivial Dixmier-Douady class and if Q is a trivialization of L^* , then we can choose $\mathcal{K} = \underline{\mathbb{C}} = L_{\mathcal{K}}$ in the above. In this case we get

$$(16) \quad \text{ch}_Q(E) = \text{ch}(Q(E)) \in H^{2*}(M, K_0(A) \otimes \mathbb{R}) ,$$

where on the right hand side the Mishchenko-Fomenko Chern Character is used. Now suppose that $\mathcal{L} = \overline{\mathcal{K}}$, i.e. the complex conjugate matrix bundle. Then $dd(L_{\mathcal{L}}) = -dd(L_{\mathcal{K}}) = -dd(L)$. A trivialization Q of $(L \boxtimes L_{\mathcal{L}})^* = L^* \boxtimes L_{\mathcal{K}}$ induces now two Chern characters, which arise from viewing Q either as a morphism between $L \boxtimes L_{\mathcal{L}}$ and the trivial bundle gerbe or as a morphism between L and $L_{\mathcal{K}}$

$$\begin{aligned} \text{ch}_Q: K_{\mathcal{L} \otimes \mathcal{A}}^0(M) &\rightarrow H^{2*}(M, K_0(A) \otimes \mathbb{R}) , \\ \text{ch}_{L_{\mathcal{L}} \boxtimes Q}: K_{\mathcal{L} \otimes \mathcal{A}}^0(M) &\rightarrow H^{2*}(M, K_0(A) \otimes \mathbb{R}) . \end{aligned}$$

Corollary 3.19. *The two Chern characters described above agree. In particular, we get for \mathcal{L} with $dd(L_{\mathcal{L}}) = -dd(L)$ and $[F] \in K_{\mathcal{L}}^0(M)$, $[E] \in K_{\mathcal{A}}^0(M)$*

$$\text{ch}(Q(F \boxtimes E)) = \text{ch}_Q(F \boxtimes E) = \text{ch}_{L_{\mathcal{L}} \boxtimes Q}(F \boxtimes E) = \text{ch}(F) \cup \text{ch}_Q(E) .$$

Proof. As in the previous paragraph we set $\mathcal{K} = \overline{\mathcal{L}}$. Note that $C(M, \mathcal{K} \otimes \mathcal{L})$ and $C(M)$ are Morita equivalent via the imprimitivity bimodule $C(M, \mathcal{K})$ and the Chern character on $K_0(C(M, \mathcal{K} \otimes \mathcal{L})) \cong K_{\mathcal{K} \otimes \mathcal{L}}^0(M)$ agrees with the one on $K^0(M)$, since the bundle gerbe $L_{\mathcal{K}} \boxtimes L_{\mathcal{L}}$ has a flat trivialization. In the following diagram

$$\begin{array}{ccccc} K_{\mathcal{L} \otimes \mathcal{A}}^0(M) & \xrightarrow{(1)} & K_0(C(M, A)) & \xrightarrow{(4)} & H^{2*}(M, K_0(A) \otimes \mathbb{R}) \\ \downarrow = & & \downarrow (3) & & \downarrow = \\ K_{\mathcal{L} \otimes \mathcal{A}}^0(M) & \xrightarrow{(2)} & K_0(C(M, \mathcal{K} \otimes \mathcal{L} \otimes A)) & \xrightarrow{(5)} & H^{2*}(M, K_0(A) \otimes \mathbb{R}) \end{array}$$

the maps (4) and (5) arise from the Künneth splitting and the Chern character on $K^0(M)$ and $K_{\mathcal{K} \otimes \mathcal{L}}^0$ respectively. (3) is given by applying the Morita equivalence described above. Thus, the right hand square commutes. Checking the Morita equivalences that induce the maps (1) and (2) it is clear that the left hand square commutes as well. The upper horizontal arrows now yield ch_Q , while the lower composition is $\text{ch}_{\mathcal{L} \boxtimes Q}$ and the corollary follows. \square

Remark 3.20. With minor changes, which are complete analogues of the non-twisted case [16], we can define twisted K -groups for non-compact manifolds and relative K -groups $K_{\mathcal{A}}^0(M, N)$ for a closed subspace $N \subset M$ as well. The latter will consist of triples $[E, F, \alpha]$, where E and F are twisted Hilbert A -module bundles over M and $\alpha \in C_b(M, \text{hom}(E, F))$ is a morphism of twisted Hilbert A -module bundles, which is an isomorphism over N . Choosing an appropriate bundle of matrix algebras \mathcal{K} over M , there still is an isomorphism $K_{\mathcal{A}}^0(M, N) \cong K_{\mathcal{K} \otimes \mathcal{A}}^0(M, N)$. In case E and F are bundle gerbe modules with $[E, F, \alpha] \in K_{\mathcal{K}}^0(M, N)$, we can define a Chern character, which takes values in the corresponding relative cohomology groups. To do so, we have to choose twisted connections ∇^E and ∇^F such that ∇^E agrees with $\alpha^{-1} \circ \nabla^F \circ \alpha$ in a neighborhood of N . This can always be achieved by an appropriate deformation of an arbitrary connection on E . Now define

$$\text{ch}([E, F, \alpha]) = \text{tr} \left(\exp \left(\frac{i \Omega_E}{2\pi} \right) \right) - \text{tr} \left(\exp \left(\frac{i \Omega_F}{2\pi} \right) \right) \in H^{2*}(M, N; \mathbb{R})$$

The Künneth theorem for K -theory then allows us to define a Chern character with values in $H^{2*}(M, N; K_0(A) \otimes \mathbb{R})$ just like above preserving all the properties, in particular its multiplicativity.

3.1.2. Chern character and traces. In many applications the C^* -algebra A under consideration comes equipped with a trace $\tau: A \rightarrow \mathbb{C}$, i.e. with a continuous positive linear functional with the trace property. τ induces a trace τ_V on the adjointable endomorphisms $\text{End}(V)$ of any finitely generated projective Hilbert A -module V by the following construction: We have

$$(17) \quad \text{End}(V) = \mathcal{K}(V) \cong V \otimes_A \mathcal{K}(V, A) ,$$

where $\mathcal{K}(V, W)$ and $\mathcal{K}(V)$ denote the compact A -linear operators. On elementary tensors we therefore define $\tau_V(v \otimes T) = \tau(T(v))$, which is easily seen to extend to a trace on $\text{End}(V)$. In the case $V = A^n$ we have $\tau_V = \text{tr} \otimes \tau$ on $\text{End}(V) = M_n(A) = M_n(\mathbb{C}) \otimes A$.

Now let E be a twisted Hilbert A -module over P with respect to a lifting bundle gerbe L . Throughout this and the next section we will assume that L has a *flat* connection ∇^L . Since $\text{end}(E)$ is associated to P via the adjoint action of Γ on $\text{End}(V)$, the trace extends to a map on $\text{end}(E)$ -valued forms, which we will also denote by τ . Thus, with a trace at hand there is a more direct approach to the Chern character. Let ∇^E be a twisted connection on E . It induces a connection ∇^* on the bundle $\text{Hom}(E, \underline{A})$ in such a way that $d(\varphi(u)) = (\nabla^* \varphi)(u) + \varphi(\nabla^E u)$ for $\varphi \in C^\infty(P, \text{Hom}(E, \underline{A}))$ and $u \in C^\infty(P, E)$. Using the isomorphism (17) we get a connection ∇ on the bundle $\text{end}(E)$ in such a way that if we interpret a section of $\text{end}(E)$ as a section of $\text{End}(E)$ over P we have $\nabla^E(\psi(u)) = \nabla(\psi)(u) + \psi(\nabla^E(u))$ for $\psi \in C^\infty(M, \text{end}(E))$. The connection ∇ can be extended to forms by demanding the graded Leibniz rule as usual.

Lemma 3.21. *If τ is the trace extended to $\text{end}(E)$ -valued forms over M , then*

$$d\tau(\omega) = \tau(\nabla\omega)$$

for an arbitrary connection ∇ on $\text{end}(E)$ induced by a twisted connection ∇^E on E as described above.

Proof. The proof of this result is contained in [30, lemma 4.1.35]: We can identify $\Omega^k(M, \text{end}(E))$ with horizontal k -forms $\Omega_{\text{hor}}^k(P, \text{End}(E))$ and restrict to elements $\omega \in \Omega_{\text{hor}}^k(P, \text{End}(E))$, which are of the form $\omega = \alpha u \otimes \psi$ for $\alpha \in \Omega_{\text{hor}}^k(P)$, $u \in C^\infty(P, E)$, $\psi \in C^\infty(P, \text{Hom}(E, \underline{A}))$ by a partition of unity argument. Now

$$\nabla(\omega) = d\alpha u \otimes \psi + (-1)^{\deg(\alpha)} \alpha (\nabla_E u \otimes \psi + u \otimes \nabla^* \psi) .$$

Taking the trace yields:

$$\begin{aligned} \tau(\nabla(\omega)) &= \tau(d\alpha \psi(u) + (-1)^{\deg(\alpha)} \alpha (\psi(\nabla_E u) + \nabla^* \psi(u))) \\ &= \tau(d\alpha \psi(u) + (-1)^{\deg(\alpha)} \alpha d(\psi(u))) = d(\tau(\alpha \psi(u))) . \end{aligned} \quad \square$$

Definition 3.22. Let M be a compact smooth manifold and E be a twisted Hilbert A -module bundle over a principal Γ -bundle $P \rightarrow M$ with a twisted connection ∇^E that has curvature $\Omega_E \in \Omega^2(M, \text{end}(E))$. It follows from [30, lemma 5.2] together with lemma 3.21 that

$$\tau \left(\exp \left(\frac{i\Omega_E}{2\pi} \right) \right) \in \Omega^{\text{even}}(M)$$

is a closed form, which is independent of the choice of twisted connection. Its cohomology class $\text{ch}_\tau(E)$ is called the τ -Chern character of E . Similar to the construction noted in remark 3.20 there is also a corresponding τ -Chern character for non-compact manifolds.

Remark 3.23. If we view bundle gerbe modules as twisted Hilbert $M_n(\mathbb{C})$ -module bundles, then the Chern character for $f = 0$ described in definition 3.15 corresponds to ch_{τ_1} , where τ_1 is the *normalized* trace on $M_n(\mathbb{C})$, i.e. $\tau_1(T) = \frac{1}{n}\text{tr}(T)$ for $T \in M_n(\mathbb{C})$.

To explain the connection between ch_Q and ch_τ we need the following concept.

Definition 3.24. Let V be a finitely generated projective right Hilbert A -module. Its *dimension* is defined to be $\dim_\tau(V) = \tau_V(\text{id}_V)$, where τ_V is the extension of τ to $\text{End}(V)$ like above. If $V \cong tA^n$ for some projection $t \in M_n(A) = M_n(\mathbb{C}) \otimes A$, then $\dim_\tau(V) = (\text{tr} \otimes \tau)(t)$. \dim_τ yields a well-defined map $K_0(A) \rightarrow \mathbb{R}$.

Theorem 3.25. Let L be a bundle gerbe with $dd(L)$ torsion, let \mathcal{K} be a bundle of matrix algebras with $dd(L_{\mathcal{K}}) = dd(L)$ and let Q be a trivialization of $L^* \boxtimes L_{\mathcal{K}}$. Then we have

$$(18) \quad \dim_\tau(\text{ch}_Q(E)) = \text{ch}_\tau(E) \cup \text{ch}(Q) \in H^{\text{even}}(M, \mathbb{R}),$$

where $\text{ch}(Q)$ is the Chern character of Q as a bundle gerbe module for $L^* \boxtimes L_{\mathcal{K}}$.

Proof. Just like in the proof of theorem 3.16 it follows that ch_τ is multiplicative with respect to the product (15), i.e. that for a twisted Hilbert A -module bundle E and a bundle gerbe module F we have

$$\text{ch}_\tau(E \boxtimes F) = \text{ch}_\tau(E) \cup \text{ch}(F).$$

Now consider the case $\mathcal{A} = \mathcal{K} \otimes A$ with trace $\tau_1 \otimes \tau$ on $M_n(\mathbb{C}) \otimes A$ and $Q = \underline{\mathbb{C}}$. If F is a bundle gerbe module for $L_{\mathcal{K}}$ and V is a finitely generated projective Hilbert A -module, then $\text{ch}_{\underline{\mathbb{C}}}(F \otimes V) = \text{ch}(F) \otimes [V] \in H^{\text{even}}(M, K_0(A) \otimes \mathbb{R})$. Observe that for the trivial bundle \underline{V} we have $\text{ch}_\tau(\underline{V}) = \dim_\tau(V)$. Thus,

$$\text{ch}_\tau(F \otimes V) = \text{ch}_\tau(F \boxtimes \underline{V}) = \text{ch}(F) \cup \text{ch}_\tau(\underline{V}) = \dim_\tau(V) \text{ch}(F).$$

Thus, it remains to be checked that ch_τ vanishes on the second summand in the Künneth decomposition. We can identify $K_{\mathcal{K}}^1(M)$ with $K_{\pi_M^* \mathcal{K}}^0(\mathbb{R} \times M)$. Likewise we can represent classes in $K^1(A)$ as compactly supported virtual Hilbert A -module bundles over \mathbb{R} . If $F = (F_+, F_-, \varphi)$ represents an element in $K_{\pi_M^* \mathcal{K}}^0(\mathbb{R} \times M)$ and $W = (W_+, W_-, \psi)$ an element in $K_0(C_0(\mathbb{R}, A))$, then the graded tensor product $F \hat{\boxtimes} W$ yields an element in $K_{\pi_M^* \mathcal{K} \otimes A}^0(\mathbb{R}^2 \times M)$ and Bott periodicity maps it to $b([F \hat{\boxtimes} W]) \in K_{\mathcal{K} \otimes A}^0(M)$. It is a consequence of the multiplicativity that $\text{ch}_\tau(X) = \text{ch}_\tau(b(X)) \cup e$, where $e \in H_c^2(\mathbb{R}^2, \mathbb{R}) \cong \mathbb{R}$ denotes a generator and X is a triple representing a class in $K_{\pi_M^* \mathcal{K} \otimes A}^0(\mathbb{R}^2 \times M)$. Thus,

$$\text{ch}_\tau(b(F \boxtimes W)) \cup e = \text{ch}_\tau(F \boxtimes W) = \text{ch}(F) \cup \text{ch}_\tau(W) = 0$$

since $\text{ch}_\tau(W) \in H_c^{\text{even}}(\mathbb{R}, \mathbb{R}) = 0$ and we have proven (18) in this special case. For the general case just note that

$$\dim_\tau(\text{ch}_Q(E)) = \dim_\tau(\text{ch}_{\underline{\mathbb{C}}}(Q(E))) = \text{ch}_\tau(Q(E)) = \text{ch}_\tau(E) \cup \text{ch}(Q)$$

since the curvature of the induced connection $\nabla^{Q(E)}$ satisfies $\Omega_{Q(E)} = \Omega_E + \Omega_Q$. \square

3.1.3. Twisted bundles and local C^* -algebras. Throughout the last sections we have rarely used the completeness of the C^* -algebra A or the corresponding modules. For calculations it is often desirable to weaken the completeness assumption. For example, we may not have a trace on A , but just on a dense subalgebra $B \subset A$ as it is the case for $A = \mathbb{K}$ and B the trace class operators. Local C^* -algebras as in [5] provide the right framework for this. In the non-twisted case the corresponding observations can be found in [30].

Definition 3.26. A *local C^* -algebra* is a $*$ -algebra B equipped with a pre- C^* -norm such that $M_n(B)$ is closed under holomorphic functional calculus for all $n \in \mathbb{N}$.

As a direct consequence of the definition we see that algebras of sections inherit the locality property from the fiber: Let B be a unital local C^* -algebra, P be a principal Γ -bundle over the compact manifold M together with a homomorphism $\Gamma \rightarrow U(B)/U(1)$. Let $\mathcal{B} = P \times_{\text{Ad}} B$ be the associated bundle of local C^* -algebras, then $C(M, \mathcal{B})$ is a local C^* -algebra as well. If A is the C^* -completion of B and $\mathcal{A} = P \times_{\text{Ad}} A$, then $C(M, \mathcal{A})$ is the C^* -completion of $C(M, \mathcal{B})$. A right B -module is called *finitely generated* and *projective* if it is isomorphic to one of the form tB^n for a projection $t \in M_n(B)$. Observe that $\text{End}(tB^n) = tM_n(B)t$ is a local C^* -algebra with completion $\text{End}(tA^n) = tM_n(A)t$. The definition of the groups $K_0(B)$ and $K_1(B)$ is now just like in [5].

This enables us to define twisted (inner product) B -module bundles replacing Hilbert A -modules in definition 3.1 by finitely generated projective inner product B -modules. Taking the Grothendieck group of the isomorphism classes of such gadgets yields an abelian group $K_B^0(M)$ and we set $K_B^1(M) = K_B^0(\mathbb{R} \times M)$ (see remark 3.20). Given a twisted B -module \mathcal{E} , the fiberwise inner product $\mathcal{E} \otimes_B A$ yields a twisted Hilbert A -module bundle. Thus we get a homomorphism

$$(19) \quad K_B^0(M) \rightarrow K_A^0(M) .$$

The proofs of lemma 3.7 and theorem 3.9 rely completely on holomorphic functional calculus for algebras of the form $\text{End}(V)$ for a finitely generated projective Hilbert A -module V and the fact, that the group of invertible elements forms an open subset inside a C^* -algebra. The latter is still true for local C^* -algebras. Thus, we gain an isomorphism

$$K_B^0(M) \cong K_0(C(M, \mathcal{B}))$$

which is natural in M . This immediately implies that (19) is an isomorphism as well. Let $L, L_{\mathcal{K}}, P_{\mathcal{K}}$ and Q be as in section 3.1. Note that the exterior tensor product $S^* \boxtimes E$ with $S = P_{\mathcal{K}} \times \mathbb{C}^N$ involves no completion at all. Therefore shifting the twist to matrix algebra bundles like in theorem 3.12 still works for twisted B -module bundles and maps \mathcal{E} to $Q(S^* \boxtimes \mathcal{E})$, i.e. there is an isomorphism

$$K_B^0(M) \cong K_{\mathcal{K} \otimes B}^0(M) \cong K_0(C(M, \mathcal{K} \otimes B)) ,$$

but note that $C(M, \mathcal{K} \otimes B)$ is in general *not* isomorphic to $C(M) \otimes B$. Nevertheless, there still is a Bott map $b: K_B^0(M) \rightarrow K_{\pi_M^* B}^0(\mathbb{R}^2 \times M)$ (with $\pi_M: \mathbb{R}^2 \times M \rightarrow M$ being the projection). It sends a twisted B -module bundle E to the triple $[\pi_M^* E, \pi_M^* E, \varphi]$, where φ is the map that multiplies each fiber with $x + iy$ for $(x, y) \in \mathbb{R}^2$ (see remark 3.20 for the definition of such triples). The commutativity of

$$\begin{array}{ccc}
K_{\mathcal{B}}^0(M) & \xrightarrow{b} & K_{\pi_M^* \mathcal{B}}^0(\mathbb{R}^2 \times M) \\
\downarrow \cong & & \downarrow \cong \\
K_{\mathcal{A}}^0(M) & \xrightarrow{\cong} & K_{\pi_M^* \mathcal{A}}^0(\mathbb{R}^2 \times M)
\end{array}$$

shows that it is an isomorphism. Likewise, there is a Künneth homomorphism

$$K_{\mathcal{K}}^0(M) \otimes K_0(B) \otimes \mathbb{R} \oplus K_{\mathcal{K}}^1(M) \otimes K_1(B) \otimes \mathbb{R} \rightarrow K_{\mathcal{K} \otimes B}^0(M) \otimes \mathbb{R}$$

that just involves algebraic tensor products in the fibers. A diagram chase similar to the one above shows that it also turns out to be an isomorphism. This finally enables us to define $\text{ch}_Q: K_{\mathcal{B}}^0(M) \rightarrow H^{\text{even}}(M, K_0(B) \otimes \mathbb{R})$, such that the following diagram commutes:

$$\begin{array}{ccc}
K_{\mathcal{B}}^0(M) & \xrightarrow{\text{ch}_Q} & H^{\text{even}}(M, K_0(B) \otimes \mathbb{R}) \\
\downarrow \cong & & \downarrow \cong \\
K_{\mathcal{A}}^0(M) & \xrightarrow{\text{ch}_Q} & H^{\text{even}}(M, K_0(A) \otimes \mathbb{R})
\end{array}$$

where the vertical maps are induced by $\mathcal{E} \rightarrow \mathcal{E} \otimes_B A$.

Suppose now that B is a unital local C^* -algebra carrying a trace τ . Let \mathcal{V} be a finitely generated and projective B -module. Because the endomorphisms of \mathcal{V} coincide with the finite rank operators, we still have

$$\text{End}(\mathcal{V}) \cong \mathcal{V} \otimes_B \text{Hom}(\mathcal{V}, B),$$

where the tensor product is a quotient of the algebraic tensor product. Thus, we can repeat the constructions in section 3.1.2 to define $\text{ch}_\tau: K_{\mathcal{B}}^0(M) \rightarrow H^{\text{even}}(M, \mathbb{R})$ and $\dim_\tau: K_0(B) \rightarrow \mathbb{R}$. Now the proof of theorem 3.25 applies to the new setting as well and we have

$$(20) \quad \dim_\tau(\text{ch}_Q(\mathcal{E})) = \text{ch}_\tau(\mathcal{E}) \cup \text{ch}(Q).$$

3.1.4. Generalized projective Dirac operators. Let E be a twisted Hilbert A -module bundle over a principal Γ -bundle $\pi: P \rightarrow M$ for a bundle gerbe L . A section of E over M should correspond to a Γ -equivariant section of $E \rightarrow P$, which does not exist exactly due to the twisting. It therefore seems impossible to give a good notion of pseudodifferential operators acting on E , although it is straightforward to give a good notion of what their symbols should be as has already been noticed in [20]. Note that $\pi^* T^* M$ embeds into $T^* P$ as those covectors, which vanish on vertical vector fields generated by the group action of Γ .

Definition 3.27. Let $D: C^\infty(P, E) \rightarrow C^\infty(P, E')$ be a first-order differential operator. If the restriction of the principal symbol $\sigma_D: T^* P \rightarrow \text{Hom}(E, E')$ of D to the subbundle $\pi^* T^* M$ is Γ -equivariant, i.e. descends to $\sigma_D: T^* M \rightarrow \text{hom}(E, E')$, then D will be called *descending*.

Definition 3.28. Let E be a $\mathbb{Z}/2\mathbb{Z}$ -graded twisted Hilbert A -module bundle with connection ∇^E such that the twisting and the connection preserve the grading. Moreover, let $\nabla^{T^* M}$ be a connection on $T^* M \rightarrow M$. A twisted bundle morphism

$$c: T^* M \rightarrow \text{end}(E)$$

will be called a *Clifford symbol* if the following conditions are satisfied:

- it takes values in the anti-self-adjoint ($c^* = -c$), odd part of $\text{end}(E)$
- it squares to the symbol of the Laplace operator, i.e.

$$c(\xi)^2 = -\|\xi\|^2 \text{id}_E$$

- it satisfies the following product rule with respect to ∇^E :

$$\nabla_{\widehat{X}}^E(c(\xi)u) = c\left(\nabla_{\pi_*\widehat{X}}^{T^*M}\xi\right)u + c(\xi)\left(\nabla_{\widehat{X}}^E u\right)$$

for a section $u: P \rightarrow E$ and $\widehat{X} \in T_p P$, $\xi \in T_{\pi(p)}^* M$.

A descending first-order differential operator will be called *generalized projective Dirac operator* if its symbol (as a map $T^*M \rightarrow \text{end}(E)$) is Clifford.

Example 3.29. Every Clifford symbol $c: T^*M \rightarrow \text{end}(E)$ gives rise to a generalized projective Dirac operator D having c as its principal symbol. To see this, choose a twisted connection ∇ on E and a connection on the principal bundle $\pi: P \rightarrow M$. The latter yields a projection $T^*P \rightarrow \pi^*T^*M$. The operator

$$D: C^\infty(P, E) \xrightarrow{\nabla} C^\infty(P, E \otimes T^*P) \rightarrow C^\infty(P, E \otimes \pi^*T^*M) \xrightarrow{c} C^\infty(P, E)$$

is descending with principal symbol c . Since the twisting preserves the grading, E splits as $E = E_+ \oplus E_-$ and the odd operator D decomposes correspondingly into

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

Suppose E is a twisted Hilbert A -module bundle over P_1 for the bundle gerbe L_1 . Take another bundle gerbe L_2 with $dd(L_1) = dd(L_2)$. Choose a trivialization Q of $L_1^* \boxtimes L_2$. Moreover, suppose L_i is equipped with a connection ∇^{L_i} and Q carries a bundle gerbe module connection for the induced one on $L_1^* \boxtimes L_2$. Note that $\text{end}(E) = \text{end}(Q(E))$. Therefore a choice of connection on the principal bundle P_2 enables us to define the transferred Dirac operator D^Q via theorem 2.29 by

$$\begin{aligned} D^Q: C^\infty(P_2, Q(E)) &\xrightarrow{\nabla^{Q(E)}} C^\infty(P_2, Q(E) \otimes T^*P_2) \rightarrow C^\infty(P_2, Q(E) \otimes \pi^*T^*M) \\ &\xrightarrow{c} C^\infty(P_2, Q(E)) \end{aligned}$$

Remark 3.30. Due to the projection from T^*P to π^*T^*M these generalized Dirac operators are not elliptic. However, they are transversally elliptic with respect to the subbundle $\pi^*T^*M \subset T^*P$ in the sense of Atiyah [3]. In the case of twisted Hilbert A -module bundles for a matrix algebra $A = M_n(\mathbb{C})$, i.e. ordinary bundle gerbe modules, these operators have been studied by Mathai in [21].

As in the untwisted case, we can twist a projective Dirac operator D on the twisted Hilbert A -module bundle E for L_1 over P_1 with a bundle gerbe module $F \rightarrow P_2$ for L_2 . Indeed, if ∇^E is a twisted connection on E and ∇^F a corresponding one on F , there is an induced connection $\nabla^{E \boxtimes F}$ on the twisted Hilbert A -module bundle $E \boxtimes F \rightarrow P_1 \times_M P_2$. Moreover, $\text{end}(E \boxtimes F) = \text{end}(E) \otimes \text{end}(F)$, therefore $c \otimes \text{id}_F$ is a well-defined Clifford symbol. Hence, replacing ∇^E by $\nabla^{E \boxtimes F}$ and c by $c \otimes \text{id}_F$ we get another projective Dirac operator, which we will denote by D^F . Likewise, we can of course just switch the roles of the twisted Hilbert A -module bundle E and the bundle gerbe module F and twist a projective Dirac operator acting on sections of F with E resulting in D^E .

Combining both procedures we can now use a countertwisting to reduce projective Dirac operators to ordinary ones as follows. Suppose D is a projective Dirac

operator acting on sections of the twisted Hilbert A -module bundle E for L_1 with $dd(L_1)$ torsion. Choose a matrix bundle \mathcal{K} with associated lifting bundle gerbe L_2 such that $dd(L_2) = -dd(L_1)$. Let n be such that $M_n(\mathbb{C})$ is the fiber of \mathcal{K} and set $S = \underline{\mathbb{C}}^n$. By the above we get

$$D^S : C^\infty(P_1 \times_M P_2, S \boxtimes E) \rightarrow C^\infty(P_1 \times_M P_2, S \boxtimes E)$$

and choosing a trivialization Q for $L_1 \boxtimes L_2$ we end up with

$$D^{S,Q} : C^\infty(M, Q^*(S \boxtimes E)) \rightarrow C^\infty(M, Q^*(S \boxtimes E))$$

which is now an ordinary generalized Dirac operator with principal symbol $\text{id}_S \otimes c$.

3.1.5. Classes in twisted K -homology. Just as ordinary non-projective Dirac operators acting on Hilbert A -module bundles yield classes in the K -homology group $KK(C(M), A)$, the projective Dirac operators for torsion twists can be used to define classes in *twisted K -homology* with coefficients in A , which we define to be $KK(C(M, \mathcal{K}), A)$ for a bundle of matrix algebras \mathcal{K} . In the case of Clifford bundles this has been studied by Murray and Singer in [24]. Before we state our construction of the twisted K -homology class associated to a generalized projective A -linear Dirac operator, we need some preliminaries from the theory of unbounded operators on Hilbert A -modules.

Definition 3.31. Let V, V' be Hilbert A -modules. A densely defined operator $T : V \rightarrow V'$ with densely defined adjoint $T^* : V' \rightarrow V$ is called *regular* if $1 + T^*T$ is surjective or equivalently if its graph $G(T)$ is orthocomplemented.

As the next lemma shows, which is proven for example in proposition 21 of [34], generalized projective Dirac operators belong to the particularly nice class of regular ones.

Theorem 3.32. *Let D be a generalized projective A -linear Dirac operator acting on sections of the twisted Hilbert A -module bundle E countertwisted by a bundle gerbe module S (as before we can switch the roles of E and F and the theorem remains valid). Let Q be a trivialization. Then, $D^{S,Q}$ extends to an unbounded, self-adjoint, regular operator on the Hilbert A -module $L^2(Q^*(S \boxtimes E))$ of square integrable sections:*

$$D^{S,Q} : L^2(Q^*(S \boxtimes E)) \longrightarrow L^2(Q^*(S \boxtimes E)) .$$

Regular operators can be turned into bounded adjointable ones by the Woronowicz- (or bounded-) transform [18]:

$$T \mapsto T(1 + T^*T)^{-\frac{1}{2}} .$$

Now fix a countertwisting S for D . By the argument given in the proof of theorem 3.12, we can equip $L^2(Q^*(S \boxtimes E))$ with a left action of $C(M, \mathcal{K})$, i.e. there is a homomorphism:

$$\varphi_{\mathcal{K}} : C(M, \mathcal{K}) \longrightarrow \text{End}(L^2(Q^*(S \boxtimes E))) .$$

which is induced by the canonical left action of $M_n(\mathbb{C})$ on S . Since we assume S to be graded trivially, $\varphi_{\mathcal{K}}$ maps into the *even* part of the endomorphisms.

Theorem 3.33. *Let D be a generalized projective Dirac operator and let S be a countertwisting for D , $\mathcal{K} = \text{end}(S)$ be its twisted endomorphism bundle and Q a*

trivialization. There is a Fredholm module representing $D^{S,Q}$ in KK -theory defined by:

$$[D^{S,Q}] = \left[L^2(Q^*(S \boxtimes E)), \varphi_K, D^{S,Q} \left(1 + (D^{S,Q})^2 \right)^{-\frac{1}{2}} \right] \in KK(C(M, \mathcal{K}), A) .$$

Proof. A Fredholm module $[H, \varphi, T] \in KK(B, A)$ has to satisfy $[T, \varphi(b)] \in \mathcal{K}(H)$, $(T^2 - 1)\varphi(b) \in \mathcal{K}(H)$ and $(T - T^*)\varphi(b) \in \mathcal{K}(H)$ for all $b \in b$. In our case

$$T = D^{S,Q} \left(1 + (D^{S,Q})^2 \right)^{-\frac{1}{2}} .$$

Since T is self-adjoint and $1 - T^2$ is the extension of $\left(1 + (D^{S,Q})^2 \right)^{-1}$, which is compact, the latter two conditions hold. Let $p \in M$ and $e_i \in T_p M$ be a orthonormal basis at p , denote by e_i^* the dual basis of $T_p^* M$. For $f \in C^\infty(M, \mathcal{K})$ the commutator at the point p is

$$[D^{S,Q}, \varphi_K(f)] = - \sum_i c_F(e_i^*) \boxtimes \varphi_K(\nabla_{e_i}^{\mathcal{K}} f) ,$$

i.e. the commutator is a bounded (zero-order) operator on $L^2(Q^*(S \boxtimes E))$. Now we apply a technique used in the classical case in [4] and explained in detail for the C^* -algebra case in [7] (see also [30, 5]). We can express T by the integral:

$$T = \frac{2}{\pi} \int_0^\infty D^{S,Q} \left((D^{S,Q})^2 + 1 + \lambda^2 \right)^{-1} d\lambda ,$$

where Tu converges in norm if $u \in H^1(Q^*(S \boxtimes E))$ – the first Sobolev space of sections. Apply $[T, S^{-1}] = -S^{-1}[T, S]S^{-1}$ to get:

$$\begin{aligned} [T, \varphi_K(f)] &= \frac{2}{\pi} \int_0^\infty K((1 + \lambda^2)[D^{S,Q}, \varphi_K(f)] - D^{S,Q}[D^{S,Q}, \varphi_K(f)]D^{S,Q}) K d\lambda \\ K &= \left((D^{S,Q})^2 + 1 + \lambda^2 \right)^{-1} . \end{aligned}$$

The bounds $\|D^{S,Q}K\| \leq C(d + \lambda^2)^{-\frac{1}{2}}$ and $\|K\| \leq (d + \lambda^2)^{-1}$ proven by Bunke in [7] for positive constants C and d show that the commutator integral actually converges in norm. Since $D^{S,Q}K$ is compact and $[D^{S,Q}, \varphi_K(f)]$ is bounded, the term under the integral sign is compact, therefore the commutator $[T, \varphi_K(f)]$ is as well. \square

The classes $[D^{S,Q}] \in KK(C(M, \mathcal{K}), A)$ and $[E^{S,Q}] \in KK(\mathbb{C}, C(M, \mathcal{K} \otimes A))$ interact very nicely with respect to Kasparov's intersection product as we are going to show in the following paragraph. First, we need to consider the case of trivial twisted bundles. Let \mathcal{A}_i for $i \in \{1, 2\}$ be two bundles of C^* -algebras like in example 3.4 with associated lifting bundle gerbes L_i for the groups Γ_i . Suppose that $dd(L_1) = -dd(L_2)$ and that both classes are torsion. Choose a bundle of matrix algebras \mathcal{K} associated to the principal $PU(n)$ -bundle P with bundle gerbe L , such that $dd(L) = dd(L_1)$. Let $S = \underline{\mathbb{C}^n}$ be the canonical bundle gerbe module for L . Choose a trivialization Q_1 for $L_1^* \boxtimes L$ and Q_2 for $L^* \boxtimes L_2^*$. Let $Q_{12} = Q_2 \circ Q_1$ be their composition (see remark 2.30). Let $\mathcal{V}_1 = Q_1(S^* \boxtimes \underline{\mathcal{A}_1})$ and $\mathcal{V}_2 = Q_2(S \boxtimes \underline{\mathcal{A}_2})$. The continuous sections $C(M, \mathcal{V}_1)$ form a $C(M, \mathcal{A}_1)$ – $C(M, \mathcal{K} \otimes \mathcal{A}_1)$ bimodule. Likewise, $C(M, \mathcal{V}_2)$ is a $C(M, \mathcal{K} \otimes \mathcal{A}_2)$ – $C(M, \mathcal{A}_2)$ bimodule. Finally, let $\mathcal{V}_{12} = Q_{12}(\underline{\mathcal{A}_1} \otimes \underline{\mathcal{A}_2})$, where we take the minimal tensor product (in fact, in all our applications one of the involved

C^* -algebras will be nuclear). The sections $C(M, \mathcal{V}_{12})$ form a $C(M, \mathcal{A}_1 \otimes \mathcal{A}_2)$ – $C(M, \mathcal{A}_1 \otimes \mathcal{A}_2)$ bimodule.

Lemma 3.34. *With \mathcal{V}_i and \mathcal{V}_{12} as above there is a Hilbert bimodule isomorphism:*

$$C(M, \mathcal{V}_1) \otimes_{C(M, \mathcal{K}) \otimes A_1} C(M, \mathcal{V}_2) \otimes A_1 \xrightarrow{\sim} C(M, \mathcal{V}_{12})$$

where the tensor product is the inner one taken over the inclusion $C(M, \mathcal{K}) \otimes A_1 \rightarrow C(M, \mathcal{K} \otimes \mathcal{A}_2) \otimes A_1$.

Proof. Note that there is an isomorphism of Hilbert $A_1 \otimes A_2$ – $A_1 \otimes A_2$ bimodules given by

$$\begin{aligned} \Psi : (\mathbb{C}^{n*} \otimes A_1) \otimes_{M_n(\mathbb{C}) \otimes A_1} (\mathbb{C}^n \otimes A_1 \otimes A_2) &\longrightarrow A_1 \otimes A_2 \\ (\xi \otimes a_1) \otimes_{M_n(\mathbb{C}) \otimes A_1} (v \otimes a'_1 \otimes a_2) &\mapsto \xi(v) a_1 a'_1 \otimes a_2 \end{aligned}$$

which is equivariant for the actions of $\widehat{\Gamma}_i$ given by $\widehat{\tau}_i$ (see example 3.4) and $U(n)$ in the following sense

$$\Psi((\widehat{\tau}_1(\widehat{g}_1)a_1 \otimes \xi \circ \widehat{T}^*) \otimes (\widehat{T}v \otimes a'_1 \otimes \widehat{\tau}_2(\widehat{g}_2)a_2)) = \xi(v) \widehat{\tau}_1(\widehat{g}_1)a_1 a'_1 \otimes \widehat{\tau}_2(\widehat{g}_2)a_2$$

for $\widehat{g}_i \in \widehat{\Gamma}_i$, $\widehat{T} \in U(n)$. Therefore Ψ intertwines the actions used to define \mathcal{V}_1 , \mathcal{V}_2 and \mathcal{V}_{12} as long as the trivializations used for the fibers of S and S^* are dual to each other. This way, we get a fiberwise bimodule map on the algebraic tensor product over $m \in M$

$$\Psi : (\mathcal{V}_1)_m \otimes_{\mathcal{K}_m \otimes A_1}^{\text{alg}} (\mathcal{V}_2)_m \otimes A_1 \rightarrow (\mathcal{V}_1 \otimes \mathcal{V}_2)_m ,$$

which induces a bimodule map on the sections via

$$\begin{aligned} C(M, \mathcal{V}_1) \otimes_{C(M, \mathcal{K}) \otimes A_1}^{\text{alg}} C(M, \mathcal{V}_2 \otimes A_1) &\rightarrow C(M, \mathcal{V}_{12}) \\ f \otimes h &\mapsto (x \mapsto \Psi(f(x) \otimes h(x))) . \end{aligned}$$

A small calculation shows that this is an isometry onto a dense subset of $C(M, \mathcal{V}_{12})$. Therefore it extends to a Hilbert bimodule isomorphism as stated. \square

The case of nontrivial projective twisted bundles is easily deduced from the previous lemma. We need the following instance of it.

Corollary 3.35. *Let $E_i \rightarrow P_i$ be finitely generated projective twisted Hilbert A_i -module bundles and S be a countertwisting for E_2 like above. Then there is an isomorphism θ of right Hilbert $A_1 \otimes A_2$ -modules:*

$$\theta : C(M, Q_1(S^* \boxtimes E_1)) \otimes_{\varphi} L^2(M, Q_2(S \boxtimes E_2)) \otimes A_1 \xrightarrow{\sim} L^2(M, Q_{12}(E_1 \boxtimes E_2)) .$$

where $\varphi : C(M, \mathcal{K}) \otimes A_1 \rightarrow \text{End}(L^2(M, Q_2(S \boxtimes E_2)) \otimes A_1)$ is given by left multiplication.

Proof. In case $E_i = \underline{A}_i$ the statement is a direct consequence of the previous lemma and the observation that

$$\begin{aligned} C(M, \mathcal{V}_2 \otimes A_1) \otimes_{C(M, A_2 \otimes A_1)} L^2(M, A_2 \otimes A_1) &= L^2(M, \mathcal{V}_2) \otimes A_1 \\ C(M, \mathcal{V}_{12}) \otimes_{C(M, A_2 \otimes A_1)} L^2(M, A_2 \otimes A_1) &= L^2(M, \mathcal{V}_{12}) \end{aligned}$$

as $C(M, \mathcal{A}_1 \otimes \mathcal{A}_2)$ – $A_1 \otimes A_2$ bimodules. In the general case we have projections $t_i \in C(M, M_{k_i}(\mathcal{A}_i))$ such that $Q_1(S^* \boxtimes E_1) \cong t_1 \mathcal{V}_1^{k_1}$ and $Q_2(S \boxtimes E_2) \cong t_2 \mathcal{V}_2^{k_2}$,

where the fiber of $t_i \mathcal{V}_i^{k_i}$ over $m \in M$ is spanned by the image of t_i acting on $\mathcal{V}_i^{k_i}$. Note that $(t_1 \otimes t_2) \mathcal{V}_{12}^{k_1, k_2} \cong Q_{12}(E_1 \boxtimes E_2)$, therefore

$$\begin{aligned} & C(M, Q_1(S^* \boxtimes E_1)) \otimes_{\varphi} L^2(M, Q_2(S \boxtimes E_2)) \otimes A_1 \\ & \cong C(M, t_1 \mathcal{V}_1^{k_1}) \otimes_{C(M, \mathcal{K}) \otimes A_1} L^2(M, t_2 \mathcal{V}_2^{k_2}) \otimes A_1 \\ & \cong (t_1 \otimes t_2) C(M, \mathcal{V}_1^{k_1}) \otimes_{C(M, \mathcal{K}) \otimes A_1} L^2(M, \mathcal{V}_2^{k_2}) \otimes A_1 \\ & \cong (t_1 \otimes t_2) L^2(M, \mathcal{V}_{12}^{k_1, k_2}) \cong L^2(M, Q_{12}(E_1 \boxtimes E_2)) . \quad \square \end{aligned}$$

Now we are able to prove the result alluded to in the introduction of this chapter.

Theorem 3.36. *Let D be a generalized projective Dirac operator acting on a bundle gerbe module $E_2 \rightarrow P_2$. Let S be a countertwisting and Q_2 be a trivialization of $S \boxtimes E_2$. If $E_1 \rightarrow P_1$ is a finitely generated, projective twisted Hilbert A_1 -module bundle such that $dd(P_1) = -dd(P_2)$ with countertwisting S^* and trivialization Q_1 , then:*

$$[E_1^{S, Q_1}] \otimes_{C(M, \mathcal{K})} [D^{S, Q_2}] \in KK(\mathbb{C}, A_1) \cong K_0(A_1)$$

is the class representing the Mishchenko-Fomenko index of $D^{E_1, Q_{12}}$, where Q_{12} is the trivialization induced by Q_1 and Q_2 . In particular, the intersection product only depends on the choices of trivializations involved and not directly on the countertwisting.

Proof. It suffices to show that the intersection product coincides with the Fredholm module:

$$(21) \quad \left[L^2(Q_{12}(E_1 \boxtimes E_2)), D^{E_1, Q_{12}} \left(1 + (D^{E_1, Q_{12}})^2 \right)^{-\frac{1}{2}} \right] .$$

The isomorphism $KK(\mathbb{C}, A_1) \xrightarrow{\sim} K_0(A_1)$ is defined by taking the Mishchenko-Fomenko index of the odd part of the operator involved, but since the latter is not changed under composition with invertible bounded operators, the index of $D_+^{E_1, Q_{12}} : H^s(M, Q_{12}(E_1 \boxtimes E_2^+)) \rightarrow H^{s-1}(M, Q_{12}(E_1 \boxtimes E_2^-))$ coincides with that deduced from (21). Set

$$F = D^{E_1, Q_{12}} \left(1 + (D^{E_1, Q_{12}})^2 \right)^{-\frac{1}{2}} , \quad F_2 = D^{S, Q_2} \left(1 + (D^{S, Q_2})^2 \right)^{-\frac{1}{2}} .$$

We prove that F is an F_2 -connection from which the above assertion will follow, since our representative of $[E_1^{S, Q_1}]$ contains the zero operator. Using the isomorphism θ from corollary 3.35, we form an operator

$$T_{u_1} : L^2(M, Q_2(S \boxtimes E_2)) \otimes A_1 \longrightarrow L^2(M, Q_{12}(E_1 \boxtimes E_2)) \quad ; \quad u_2 \mapsto \theta(u_1 \otimes u_2)$$

for every $u_1 \in C(M, Q_1(E_1 \boxtimes S^*))$. By definition F is an F_2 -connection if

$$T_{u_1} \circ F_2 - F \circ T_{u_1} \in \text{Hom}(L^2(M, Q_2(S \boxtimes E_2)) \otimes A_1, L^2(M, Q_{12}(E_1 \boxtimes E_2)))$$

is a compact A_1 -linear operator. It suffices to show this for smooth sections $u_1 \in C^\infty(M, Q_1(E_1 \boxtimes S^*))$, since the latter is a norm-dense subspace of the continuous sections, the operator $u_1 \mapsto T_{u_1}$ is norm-continuous and compact operators are norm-closed. For $u_1 \in C^\infty(M, Q_1(E_1 \boxtimes S^*))$ we have the following commutator identity, which holds (at least) on $C^\infty(M, Q_2(S \boxtimes E_2) \otimes A_1)$

$$(22) \quad T_{u_1} \circ D^{S, Q_2} - D^{E_1, Q_{12}} \circ T_{u_1} = - \sum_i T_{f_i} c(e_i^*) =: -S$$

with $f_i = \nabla_{e_i}^{Q_1(E_1 \boxtimes S^*)} u_1$, where c denotes the symbol of D . The right hand side of (22) is a bounded operator. Now the proof runs along the same lines as the one given in theorem 6.22 in [30]. Set

$$K_1(\lambda) = \left((D^{E_1, Q_{12}})^2 + 1 + \lambda^2 \right)^{-1}, \quad K_2(\lambda) = \left((D^{S, Q_2})^2 + 1 + \lambda^2 \right)^{-1}.$$

Replacing F and F_2 by their integral representation introduced in theorem 3.33, we have that up to compact operators:

$$\begin{aligned} & T_{u_1} \circ F_2 - F \circ T_{u_1} \\ \equiv & \int_0^\infty (T_{u_1} K_2(\lambda) D^{S, Q_2} - K_1(\lambda) T_{u_1} D^{S, Q_2}) d\lambda \\ = & \int_0^\infty K_1(\lambda) \left((D^{E_1, Q_{12}})^2 T_{u_1} - T_{u_1} (D^{S, Q_2})^2 \right) K_2(\lambda) D^{S, Q_2} d\lambda \\ = & \int_0^\infty K_1(\lambda) (D^{E_1, Q_{12}} S + S D^{S, Q_2}) K_2(\lambda) D^{S, Q_2} d\lambda \end{aligned}$$

Since $K_1(\lambda)$, $K_1(\lambda) D^{E_1, Q_{12}}$ and $K_2(\lambda) D^{S, Q_2}$ are compact and $D^{S, Q_2} K_2(\lambda) D^{S, Q_2}$ is bounded, the integrand is compact. The estimates

$$\begin{aligned} \|K_1(\lambda)\| &\leq (d_1 + \lambda^2)^{-1}, \quad \|K_1(\lambda) D^{E_1, Q_{12}}\| \leq C_1 (d_1 + \lambda^2)^{-\frac{1}{2}} \\ \|K_2(\lambda) D^{S, Q_2}\| &\leq C_2 (d_2 + \lambda^2)^{-\frac{1}{2}}, \quad \|K_2(\lambda) (D^{S, Q_2})^2\| \leq C_2 \end{aligned}$$

proven by Bunke in [7] yield the convergence of the integral in norm, finally showing that the commutator is a compact operator. \square

Remark 3.37. We could of course rephrase the above theorem for D acting on sections of a twisted Hilbert A_2 -module bundle and some bundle gerbe E_1 or even for two twisted Hilbert A_i -module bundles, where the index then would take values in $K_0(A_1 \otimes A_2)$, but we won't need this in the applications.

The decomposition of the index in twisted K -theory directly yields a nice proof of the following naturality result, which will play an important role in the application presented in the next chapter (compare with the untwisted case presented in lemma 3.1 in [13]).

Corollary 3.38. *Let D be a generalized projective Dirac operator acting on a bundle gerbe module E_2 and let E_1 be a twisted Hilbert A_1 -module bundle. Given a C^* -algebra homomorphism $\varphi : A_1 \rightarrow B_1$ define the twisted Hilbert B_1 -module bundle F_1 via*

$$F_1 = E_1 \otimes_\varphi B_1.$$

Then: $\varphi_([D^{E_1, Q_{12}}]) = [D^{F_1, Q_{12}}]$, where $\varphi_* : K_0(A_1) \rightarrow K_0(B_1)$ denotes the induced map on K -theory and Q_{12} is a trivialization.*

Proof. Choose a countertwisting S and trivializations Q_i , such that $Q_{12} = Q_2 \circ Q_1$ (see remark 2.31). Applying φ_* to $[E_1^{S, Q_1}]$ yields

$$\begin{aligned} (\text{id}_{C(M, \mathcal{K})} \otimes \varphi_*)[E_1^{S, Q_1}] &= [C(M, Q_1(S^* \boxtimes E_1)) \otimes_{\text{id} \otimes \varphi} B_1, 0] \\ &= [C(M, Q_1(S^* \boxtimes (E_1 \otimes_\varphi B_1))), 0] = [F_1^{S, Q_1}]. \end{aligned}$$

Therefore, by naturality of the Kasparov product, we get

$$\begin{aligned}\varphi_*([D^{E_1, Q_{12}}]) &= \varphi_*([E_1^{S, Q_1}] \otimes_{C(M, \mathcal{K})} [D^{S, Q_2}]) \\ &= ((\text{id}_{C(M, \mathcal{K})} \otimes \varphi_*)[E_1^{S, Q_1}]) \otimes_{C(M, \mathcal{K})} [D^{S, Q_2}] \\ &= [F_1^{S, Q_1}] \otimes_{C(M, \mathcal{K})} [D^{S, Q_2}] = [D^{F_1, Q_{12}}] . \quad \square\end{aligned}$$

4. INDEX OBSTRUCTIONS AGAINST PSC METRICS

In this section we will decompose the index invariant developed by Stolz for manifolds with a spin structure on the universal cover [31] into a pairing of a twisted K -theory with a twisted K -homology class. Let (M, g) be a closed orientable Riemannian manifold of even dimension with Levi-Civita-connection $\nabla = \nabla^{TM}$ and a spin structure on the universal cover. The Riemannian curvature transformation is defined to be

$$R \in \Omega^2(M, \text{End}(TM)) , \quad R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U, V]} W = \nabla \circ \nabla ,$$

where the second definition uses the obvious extension of the covariant derivative to forms via the Leibniz rule. The simplest invariant obtained from R is the scalar curvature κ defined by

$$\kappa(m) = - \sum_{i, j=1}^n \langle R(e_i, e_j)e_i, e_j \rangle ,$$

where the e_i form an orthonormal basis of $T_m M$ – a choice, on which the value of κ does not depend. Let $P = P_{SO}$ be the bundle of oriented frames in TM and denote by L_{spin} the lifting bundle gerbe associated to the central S^1 -extension

$$1 \rightarrow S^1 \rightarrow \text{Spin}^c(n) \rightarrow SO(n) \rightarrow 1$$

like in example 2.3. Let $S \rightarrow P$ be the $\mathbb{Z}/2\mathbb{Z}$ -graded spinor module for L_{spin} from example 2.17. Let $\eta_{SO} \in \Omega^1(P, \mathfrak{so}(n))$ be the connection form of ∇^{TM} . The latter induces a twisted connection ∇^S on S as described in example 2.24. Note that $\text{end}(S)$ coincides with the complex Clifford bundle $\mathbb{C}\ell(M)$ over M . Let $f: M \rightarrow TM$ be a vector field. It acts on a section σ of S via Clifford multiplication $C^\infty(M, \mathbb{C}\ell(M)) \times C^\infty(P, S) \rightarrow C^\infty(P, S)$, which will be denoted by a dot. ∇^S satisfies the following Leibniz rule

$$\nabla^S(f \cdot \sigma) = (\nabla^{TM} f) \cdot \sigma + f \cdot \nabla^S \sigma .$$

Thus, there is a Clifford symbol

$$c: T^*M \xrightarrow{g} TM \longrightarrow \mathbb{C}\ell(M) = \text{end}(S)$$

in the sense of definition 3.28 and a projective Dirac operator $D: C^\infty(P, S) \rightarrow C^\infty(P, S)$ (using η_{SO} as the connection on P). Let $p \in P$ with $\pi(p) = m \in M$ and let $e_i \in T_m M$ be a local orthonormal frame, then D takes the form

$$(D\sigma)(p) = \sum_i e_i \cdot \nabla_{\hat{e}_i} \sigma(p) ,$$

where $\hat{e}_i \in T_p P$ is the horizontal lift of e_i to p via η_{SO} . The tangent space TP can be identified with $\pi^* TM \oplus \mathfrak{so}(n)$ via η_{SO} . This bundle comes equipped with a natural connection ∇^P given by the pullback of ∇^{TM} plus the flat connection on

the trivial vector bundle $\underline{\mathfrak{so}(n)}$. As we will see in the next lemma this connection can have torsion.

Lemma 4.1. *Let $v, w \in C^\infty(M, TM)$ and denote their horizontal lifts via η_{SO} by $\hat{v}, \hat{w} \in C^\infty(P, TP)$, then*

$$\nabla_{\hat{v}}^P \hat{w} - \nabla_{\hat{w}}^P \hat{v} = [\hat{v}, \hat{w}] + \alpha_* \Omega_{TM}(v, w) ,$$

where $\alpha_*: P \times \mathfrak{so}(n) \rightarrow TP$ is the trivialization of the subbundle of vertical vector fields.

Proof. From the definition of ∇^P and the fact that ∇^{TM} is torsion-free we deduce that $\nabla_{\hat{v}}^P \hat{w} - \nabla_{\hat{w}}^P \hat{v} = \widehat{[v, w]}$. The lemma now follows from

$$[\hat{v}, \hat{w}] = \widehat{[v, w]} + \alpha_* \eta_{SO}([\hat{v}, \hat{w}]) = \widehat{[v, w]} - \alpha_* \Omega_{TM}(v, w) . \quad \square$$

As we will see, D^2 will contain terms based on second derivatives, which we will now define.

Definition 4.2. Let $V, W \in C^\infty(P, TP)$ be vector fields on P . The *second derivative* of a section $\sigma \in C^\infty(P, S)$ is defined by

$$\nabla_{V, W}^2 \sigma = \nabla_V \nabla_W \sigma - \nabla_{\nabla_V^P W} \sigma .$$

The *projective connection Laplacian* is the operator

$$\nabla^* \nabla_{\text{proj}}: C^\infty(P, S) \rightarrow C^\infty(P, S) \quad , \quad \nabla^* \nabla_{\text{proj}} = - \sum_i \nabla_{\hat{e}_i}^2 ,$$

where the \hat{e}_i denote a horizontal lift of an orthonormal basis of $T_m M$ at a point $m \in M$. It is easy to see that $\nabla^* \nabla_{\text{proj}}$ does not depend on this choice. It is a transversally elliptic, non-negative formally self-adjoint operator with symbol $(p, \xi) \mapsto \|\pi_* \xi\|^2$ for $(p, \xi) \in T^*P$.

The next theorem is the generalization of the formula proven by Bochner and Lichnerowicz for the square of the Dirac operator, in which the additional torsion term will enter as the operator

$$\mathfrak{T} = -\frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \nabla_{\alpha_*(\Omega_{TM}(e_i, e_j))}$$

Theorem 4.3. *Let $D: C^\infty(P, S) \rightarrow C^\infty(P, S)$ be the projective Dirac operator over the frame bundle of an even-dimensional manifold M . Then*

$$D^2 = \nabla^* \nabla_{\text{proj}} + \frac{1}{4} \kappa \circ \pi + \mathfrak{T} ,$$

where $\pi: P \rightarrow M$ denotes the projection. Let $E \rightarrow Y$ be a flat twisted Hilbert A -module bundle for L' with $dd(L') = -dd(L)$ for $L = L_{\text{spin}}$ and Q be a flat trivialization of $L \boxtimes L'$, then

$$(D^{E, Q})^2 = \nabla^* \nabla + \frac{1}{4} \kappa .$$

Proof. Fix a point $p \in P$ projecting to $m \in M$ and choose a local orthonormal frame field $e_i \in T_m M$ such that $\nabla_{e_i}^{TM} e_j = 0$ at $m \in M$. Observe that with this choice we also have $\nabla_{\hat{e}_i}^P \hat{e}_j = 0$ at p . Let $\Omega_S \in \Omega^2(M, \text{end}(S))$ be the curvature

of ∇^S , which by remark 2.23 descends to an $\text{end}(S)$ -valued 2-form on M . The difference of the second derivatives turns out to be

$$\begin{aligned}\nabla_{\hat{e}_i, \hat{e}_j}^2 - \nabla_{\hat{e}_j, \hat{e}_i}^2 &= \nabla_{\hat{e}_i} \nabla_{\hat{e}_j} - \nabla_{\hat{e}_j} \nabla_{\hat{e}_i} - \nabla \left(\nabla_{\hat{e}_i}^P \hat{e}_j - \nabla_{\hat{e}_j}^P \hat{e}_i \right) \\ &= \nabla_{\hat{e}_i} \nabla_{\hat{e}_j} - \nabla_{\hat{e}_j} \nabla_{\hat{e}_i} - \nabla[\hat{e}_i, \hat{e}_j] - \nabla_{\alpha_* \Omega_{TM}}(e_i, e_j) \\ &= \Omega_S(e_i, e_j) - \nabla_{\alpha_* \Omega_{TM}}(e_i, e_j) .\end{aligned}$$

A closer look at D^2 yields

$$\begin{aligned}D^2 &= \sum_{i,j} e_i \cdot e_j \cdot \nabla_{\hat{e}_i} \nabla_{\hat{e}_j} = - \sum_i \nabla_{\hat{e}_i, \hat{e}_i}^2 + \sum_{i < j} e_i \cdot e_j \cdot \left(\nabla_{\hat{e}_i, \hat{e}_j}^2 - \nabla_{\hat{e}_j, \hat{e}_i}^2 \right) \\ &= \nabla^* \nabla_{\text{proj}} + \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \Omega_S(e_i, e_j) + \mathfrak{T}\end{aligned}$$

The proof given in [19, Theorem 4.15] shows that the curvature of the connection on the projective spinor bundle is related to the curvature of the Levi-Civita connection in the following way

$$\Omega_S(V, W) = \frac{1}{2} \sum_{k < l} \langle \Omega_{TM}(V, W) e_k, e_l \rangle e_k \cdot e_l .$$

Plugging this into the above calculation and applying the Bianchi identities shows that

$$\mathfrak{R} = \frac{1}{2} \sum_{i,j} e_i \cdot e_j \cdot \Omega_S(e_i, e_j) = \frac{1}{4} \kappa .$$

Details for this last part can be found in [19, Theorem 8.10]. To see the second statement, observe that the operator \mathfrak{T} only contains covariant derivatives in the vertical direction. Note that sections of $Q^*(S \boxtimes E)$ can be identified those sections of $(S \boxtimes E) \otimes Q^* \rightarrow P \times_M Y$ which satisfy (9) with ϕ induced by the trivialization. As we have seen in the proof of lemma 2.28, the covariant derivative in vertical directions vanishes on these. Since we assumed Q and E to be flat, the curvature remains untouched, i.e. $\Omega_{Q^*(S \boxtimes E)} = \Omega_S$. \square

Theorem 4.4. *Let $D_+ : C^\infty(P, S_+) \rightarrow C^\infty(P, S_-)$ be the positive part of the $\mathbb{Z}/2\mathbb{Z}$ -graded projective Dirac operator D . Let $L = L_{\text{spin}}$, let L' be a bundle gerbe with $dd(L') = -dd(L)$ and let Q be a flat trivialization of $L \boxtimes L'$. Let $E \rightarrow Y$ be a twisted Hilbert A -module bundle for L' . Then we have*

$$\text{ind}(D_+^{E,Q}) = \int_M \hat{A}(M) \text{ch}_Q(E) .$$

In particular, if E is a flat bundle gerbe module for L' , we get

$$\text{ind}(D_+^{E,Q}) = N \cdot \int_M \hat{A}(M) \in \mathbb{Z} ,$$

where $N \in \mathbb{N}$ is the rank of E .

Proof. Let $\pi_M : T^*M \rightarrow M$ be the bundle projection and denote by $D(T^*M)$ and $S(T^*M)$ the disc and sphere bundle of the cotangent bundle. The symbol $\sigma : \pi_M^* Q(S_+ \boxtimes E) \rightarrow \pi_M^* Q(S_- \boxtimes E)$ of $D_+^{E,Q}$ yields an element $[Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma] \in K_A^0(D(T^*M), S(T^*M)) \cong K_A^0(T^*M)$. Denote by

$$\phi : H^{2*}(M, \mathbb{R}) \rightarrow H^{2*}(D(T^*M), S(T^*M); \mathbb{R})$$

the Thom isomorphism. From the Mishchenko-Fomenko index theorem and the multiplicativity of the twisted Chern character (see corollary 3.19 and remark 3.20) we obtain

$$\begin{aligned} \text{ind}(D_+^{E,Q}) &= \int_M \text{Td}(M) \phi^{-1} (\text{ch}([Q(S_+ \boxtimes E), Q(S_- \boxtimes E), \sigma])) \\ &= \int_M \text{Td}(M) \phi^{-1} (\text{ch}([S_+, S_-, \sigma_D])) \text{ch}_Q(E) \\ &= \int_M \hat{A}(M) \text{ch}_Q(E) \end{aligned}$$

where the identification of $\text{Td}(M) \phi^{-1} (\text{ch}([S_+, S_-, \sigma_D]))$ with the \hat{A} -genus described in [19] works in the twisted case as well. The second statement is clear from the fact that for a flat bundle gerbe module E and a flat trivialization Q we have $\text{ch}_Q(E) = \text{rank}(E) \cdot 1$. \square

4.1. Flat countertwisting. Theorem 4.3 suggests to look for flat countertwistings for the projective Dirac operator. A vector bundle is associated to a principal $U(n)$ -bundle called its frame bundle. So the question arises, whether something similar is true for a bundle gerbe module F . In fact, we can change the bundle gerbe L inside its stable isomorphism class to a lifting bundle gerbe L_F in such a way that there exist a canonical trivialization Q_F from L_F to L such that $Q_F^*(F)$ is of the form described in example 2.17.

Lemma 4.5. *Let $F \rightarrow Y$ be a bundle gerbe module for a bundle gerbe $L \rightarrow Y^{[2]}$. Let $P_F \rightarrow Y$ be its frame bundle. Then its projectivization P_F/S^1 descends to a principal $PU(n)$ -bundle $R \rightarrow M$. The associated lifting bundle gerbe L_F is stably isomorphic to L .*

Proof. The action of L on F induces a corresponding map of frame bundles, which after forming the quotient by the S^1 -action boils down to descend data

$$\bar{\gamma}: \pi_2^*(P_F/S^1) \rightarrow \pi_1^*(P_F/S^1) .$$

Therefore P_F/S^1 descends to a principal $PU(n)$ -bundle $R \rightarrow M$ and P_F/S^1 may be identified with $\pi_Y^* R = R \times_M Y$. Since $\pi_Y^* R$ lifts to the principal $U(n)$ -bundle P_F , its lifting bundle gerbe $L' \rightarrow (P_F/S^1)^{[2]}$ is trivial. In fact the line bundle Q_F associated to $P_F \rightarrow P_F/S^1$ over P_F provides a trivialization with

$$(Q_F)_{(q_1, y)} \otimes (L')_{((q_1, y), (q_2, y))} \cong (Q_F)_{(q_2, y)} \quad \text{for } ((q_1, y), (q_2, y)) \in (R \times_M Y)^{[2]} .$$

Let $L_F \rightarrow R^{[2]}$ be the lifting bundle gerbe of $R \rightarrow M$. The pullback $\pi_{Y^{[2]}}^* L_F$ coincides with L' . The action of L on E induces an action of $\pi_{R^{[2]}}^* L$ on Q_F with

$$L_{(y_1, y_2)} \otimes (Q_F)_{(q, y_2)} \rightarrow (Q_F)_{(q, y_1)} \quad \text{for } ((q, y_1), (q, y_2)) \in (R \times_M Y)^{[2]} .$$

Putting things together we end up with Q_F being a trivialization of $(L_F)^* \boxtimes L$ showing that $dd(L_F) = dd(L)$. \square

Definition 4.6. The lifting bundle gerbe L_F associated to F as in lemma 4.5 will be called the *frame bundle gerbe*, Q_F will be called the *frame trivialization*.

Lemma 4.7. *Let Q_F be the frame trivialization of a bundle gerbe module $F \rightarrow Y$ of rank n , then we have*

$$Q_F^*(F) \cong R \times \mathbb{C}^n$$

as a bundle gerbe module, where $R \times \mathbb{C}^n$ has the module structure described in example 2.17 for the canonical representation of $U(n)$ on \mathbb{C}^n .

Proof. The descent data for $Q_F^* \otimes \pi_Y^* F$ is given by the isomorphism

$$(Q_F)_{(q,y_2)}^* \otimes F_{y_2} \rightarrow (Q_F)_{(q,y_1)}^* \otimes L_{(y_1,y_2)} \otimes F_{y_2} \rightarrow (Q_F)_{(q,y_1)}^* \otimes F_{y_1}$$

for $((q,y_1), (q,y_2)) \in (R \times_M Y)^{[2]}$. Note that Q_F is the canonical line bundle over P_F/S^1 . A point $\hat{q} \in P_F$ over $y \in Y$ may be interpreted as an isomorphism $\hat{q}: \mathbb{C}^n \rightarrow E_y$. Now $(Q_F)_{(q,y_2)}^* \otimes F_{y_2}$ is canonically isomorphic to \mathbb{C}^n via

$$\begin{aligned} (Q_F)_{(q,y_2)}^* \otimes F_{y_2} &\rightarrow \mathbb{C}^n & ; & \quad [\hat{q}, \lambda] \otimes v \mapsto \lambda \hat{q}^{-1}(v) , \\ \mathbb{C}^n &\rightarrow (Q_F)_{(q,y_2)}^* \otimes F_{y_2} & ; & \quad w \mapsto [\hat{q}, 1] \otimes \hat{q}(w) , \end{aligned}$$

where \hat{q} in the second map is an arbitrarily chosen lift of $q \in R$. The actions of L on E and on Q_F^* are induced by the same map, so it is easy to see that the above isomorphism is compatible with the descent map. Since the action of L_F on Q_F^* is given by preconcatenation with elements of $U(n)$, the map $Q_F^*(F) \cong R \times \mathbb{C}^n$ respects the module structure. \square

The holonomy bundle of a flat vector bundle has a discrete structure group. Now that we have an analogue of the frame bundle for modules over bundle gerbes, we can formulate a similar statement in the twisted case.

Theorem 4.8. *Let F be a bundle gerbe module for a bundle gerbe L . Then F carries a flat connection if and only if its frame bundle gerbe $L_F \rightarrow P_F^{[2]}$ reduces to a covering bundle gerbe and the frame trivialization Q_F is flat.*

Proof. Suppose F carries a flat connection with connection form $\eta_F \in \Omega^1(P_F, \mathfrak{u}(n))$. Due to the splitting $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus i\mathbb{R}$ the projection $\mathfrak{u}(n) \rightarrow i\mathbb{R}$ induces a connection form on the line bundle Q_F , which turns out to be a bundle gerbe module connection for $(L_F)^* \boxtimes L$. By assumption it is flat. Let R be as in lemma 4.5. The frame bundle $R \times U(n)$ of $Q_F^*(F)$ inherits a flat connection form $\eta_R \in \Omega^1(R \times U(n), \mathfrak{u}(n))$ from F . Let $\iota_1: R \rightarrow R \times U(n)$ be given by $r \mapsto (r, 1)$ and let $q: U(n) \rightarrow PU(n)$ be the canonical projection. Since η_R is the connection form of a bundle gerbe module connection, $\eta = q_* \iota_1^* \eta_R \in \Omega^1(R, \mathfrak{su}(n))$ is a connection form on R . Indeed, for $V \in T_r R$, $Y \in \mathfrak{su}(n) \subset \mathfrak{u}(n)$, $a \in PU(n)$ and $\alpha_r: PU(n) \rightarrow R$ with $\alpha_r(a) = ra$ we have

$$\begin{aligned} \eta(R_{a*}V) &= q_* \eta_R(R_{a*}V, 0) = q_* \eta_R(V, L_{a*}(0)) = \text{Ad}_{a^{-1}} q_* \eta_R(V, 0) = \text{Ad}_{a^{-1}} \eta(V) , \\ \eta(\alpha_{r*}Y) &= q_* \eta_R(\alpha_{r*}Y, 0) = q_* \eta_R(0, Y) = q_* Y = Y . \end{aligned}$$

Now, η is a flat connection on R and by classical differential geometry, R reduces to the holonomy subbundle \bar{M} of η , which is a regular covering of M with classifying subgroup $\bar{\pi} = \pi_1(\bar{M}) \subset \pi_1(M)$ and deck transformation group $D = \pi_1(M)/\bar{\pi}$. Thus, $R = \bar{M} \times_{\bar{\rho}} PU(n)$ for a representation $\bar{\rho}: D \rightarrow PU(n)$ and L_F reduces to the covering bundle gerbe $\bar{L} \rightarrow \bar{M}^{[2]}$ induced by the S^1 -extension of D pulled back from $U(n) \rightarrow PU(n)$.

Now suppose that the frame bundle gerbe L_F reduces to a covering bundle gerbe $\bar{L} \rightarrow \bar{M}^{[2]}$ for some covering $\bar{M} \rightarrow M$ and that the frame trivialization Q_F is flat. The connection form given in example 2.24 reduces to the pullback of the Maurer-Cartan connection and is therefore flat. Transferring it to F via the flat line bundle Q_F yields a flat connection there as well. \square

As theorem 4.8 shows, twisted flat connections are essentially given by projective representations of the fundamental group with respect to a cocycle given by the associated covering bundle gerbe. As a matter of fact, we can retrieve the holonomy representation directly from the twisted bundle and we will need this for the general case of twisted Hilbert A -module bundles. Let $E \rightarrow P$ be a twisted Hilbert A -module bundle for the lifting bundle gerbe of a flat central S^1 -extension $\widehat{\Gamma} \rightarrow \Gamma$ and let E be equipped with a connection ∇^E . Denote by $c: [0, 1] \rightarrow P$ a smooth curve in P with $c(0) = p$. The *parallel transport* with respect to ∇^E is defined as

$$\mathcal{P}_c: [0, 1] \times E_p \rightarrow E$$

with $\mathcal{P}_c(t, v) \in E_{c(t)}$, $\mathcal{P}_c(0, \cdot) = \text{id}_{E_p}$ and $\nabla_c^E \mathcal{P}_c(\cdot, v) = 0$. Just as in classical differential geometry parallel transport exists for twisted Hilbert A -module bundles. Moreover, if ∇^E is flat, it only depends on the homotopy type of c relative endpoints. There is an action of $\widehat{\Gamma}$ on E via $\widehat{g} \cdot v = \gamma([\widehat{g}, 1] \otimes v)$, where we used the notation of example 3.4. It covers the action of Γ on P . Since it is parallel with respect to the flat connection on L and the connection on E , parallel transport is equivariant in the following sense

$$(23) \quad \mathcal{P}_{c \cdot q(\widehat{g}^{-1})}(t, \widehat{g} \cdot v) = \widehat{g} \cdot \mathcal{P}_c(t, v) .$$

Now suppose that $P = \bar{M}$ is a covering and L is a covering bundle gerbe associated to the extension $\widehat{\pi} \rightarrow \pi$ as in example 2.18, denote the action of L on E by γ^g for $g \in \pi$ and fix a basepoint $\bar{m} \in \bar{M}$ mapping to the basepoint of M . Let $\tau: S^1 \rightarrow M$ be a loop in M representing an element $h = [\tau] \in \pi_1(M)$ and denote by $\bar{\tau}$ the lift of τ to \bar{m} . Remember that the endpoint of $\bar{\tau}$ is the point $\bar{m}h^{-1}$, since we insisted on a *right* action of the deck transformation group. We abbreviate $\mathcal{P}(\bar{m}, \tau)(v) = \mathcal{P}_{\bar{\tau}}(1, v)$, where $\bar{\tau}$ is the lift of τ to $\bar{m} \in \bar{M}$, and define the holonomy around the loop τ to be $\text{hol}(\tau, \bar{m}): E_{\bar{m}} \rightarrow E_{\bar{m}}$ with

$$\text{hol}(\bar{m}, \tau) = \gamma^{h^{-1}} \circ \mathcal{P}(\bar{m}, \tau)$$

Since parallel transport and the action of L are A -linear, $\text{hol}(\bar{m}, \tau)$ is A -linear. This yields indeed a projective representation of the fundamental group. To see this, let $\sigma: S^1 \rightarrow M$ be another loop representing $g \in \pi_1(M)$ and note that the equivariance (23) implies

$$\gamma^g \circ \mathcal{P}(\bar{m}, \tau) \circ \gamma^{g^{-1}} = \mathcal{P}(\bar{m}g^{-1}, \tau) .$$

If we denote the concatenation of the two paths by $\sigma * \tau$, where we first run through σ , we get

$$\begin{aligned} c_{\bar{\pi}}(g, h)^{-1} \text{hol}(\bar{m}, \sigma * \tau) &= c_{\bar{\pi}}(g, h)^{-1} \gamma^{(gh)^{-1}} \mathcal{P}(\bar{m}, \sigma * \tau) \\ &= \gamma^{h^{-1}} \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \left(\gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}g^{-1}, \tau) \circ \gamma^g \right) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \gamma^{h^{-1}} \circ \mathcal{P}(\bar{m}, \tau) \circ \gamma^{g^{-1}} \circ \mathcal{P}(\bar{m}, \sigma) \\ &= \text{hol}(\bar{m}, \tau) \circ \text{hol}(\bar{m}, \sigma) . \end{aligned}$$

Thus, $\text{hol}(\bar{m}, \tau)$ yields a projective representation of the fundamental group.

4.1.1. *The twisted Mishchenko-Fomenko bundle.* Instead of flat twisted vector bundles, we can twist the projective Dirac operator with flat twisted Hilbert A -module bundles whenever we have a projective representation of the fundamental group $\pi = \pi_1(M)$, such that its cocycle $c_{\hat{\pi}} \in H^3(B\pi, \mathbb{Z})$ pulls back to $W_3(M) \in H^3(M, \mathbb{Z})$. Given a group π and a cocycle $c_{\hat{\pi}} \in H_{\text{gr}}^2(\pi, S^1)$, the construction of group C^* -algebras associated to π generalize to twisted group C^* -algebras.

Definition 4.9. Let π be a discrete group, $c_{\hat{\pi}} \in H_{\text{gr}}^2(\pi, S^1)$ a group 2-cocycle classifying the extension

$$1 \rightarrow S^1 \rightarrow \hat{\pi} \rightarrow \pi \rightarrow 1 .$$

Denote the associated twisted group ring by $\mathbb{C}[\pi, c_{\hat{\pi}}]$. This becomes a $*$ -algebra with respect to the involution

$$\left(\sum_{g \in \pi} \lambda_g g \right)^* = \sum_{g \in \pi} \overline{\lambda_{g^{-1}}} c_{\hat{\pi}}(g, g^{-1})^{-1} g .$$

Let $L^2(\pi)$ be the Hilbert space of elements, such that $\sum_{g \in \pi} |\lambda_g|^2 < \infty$ with the obvious scalar product. There is a twisted action of $\mathbb{C}[\pi, c_{\hat{\pi}}]$ on $L^2(\pi)$ induced by $h \cdot \lambda_g g = c_{\hat{\pi}}(h, g) \lambda_g hg$ and the above involution coincides with taking adjoints with respect to the scalar product. The closure of $\mathbb{C}[\pi, c_{\hat{\pi}}]$ with respect to the operator norm on $L^2(\pi)$ is called the *reduced twisted group C^* -algebra* $C_r^*(\pi, c_{\hat{\pi}})$.

There is another norm the twisted group algebra can be endowed with, defined by:

$$\|g\| = \sup_{\varrho} \|\varrho(g)\| ,$$

where the supremum is taken over all projective non-degenerate $*$ -representations on Hilbert spaces corresponding to the lifting cocycle $c_{\hat{\pi}}$. Since $\|\varrho(g)\|$ is bounded by the l^1 -norm, which follows from the triangle inequality, the supremum exists. The closure with respect to this norm is called the *universal (or maximal) twisted group C^* -algebra* $C_{\text{max}}^*(\pi, c_{\hat{\pi}})$. By construction it has the universal property, that any $*$ -homomorphism from $\mathbb{C}[\pi, c_{\hat{\pi}}]$ to some $B(H)$ for a Hilbert space H factors through the inclusion $\mathbb{C}[\pi, c_{\hat{\pi}}] \rightarrow C_{\text{max}}^*(\pi, c_{\hat{\pi}})$.

Apart from twisted group C^* -algebras, we will need the following canonical covering bundle gerbe associated to any orientable manifold M with $\dim(M) \geq 3$, such that the universal cover \widetilde{M} carries a spin structure: Denote by P_{SO} the oriented frame bundle of M and let $\pi = \pi_1(M)$. Since \widetilde{M} is spin, we have the following short exact sequence as was pointed out in example 2.38:

$$(24) \quad 1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \pi_1(P_{SO}) \rightarrow \pi \rightarrow 1 .$$

We set $\hat{\pi} = (\pi_1(P_{SO}) \times S^1)/(\mathbb{Z}/2\mathbb{Z})$, which is a central S^1 -extension of π . Moreover, we define $\text{Spin}^c(n) \otimes \hat{\pi}$ to be the product of the two groups modulo the diagonal S^1 -subgroup, which fits into a short exact sequence

$$(25) \quad 1 \rightarrow \text{Spin}^c(n) \rightarrow \text{Spin}^c(n) \otimes \hat{\pi} \rightarrow \pi \rightarrow 1 .$$

Definition 4.10. The covering bundle gerbe $L_{\hat{\pi}}$ associated to the central extension (24) as explained in example 2.4 is called the *Mishchenko-Fomenko bundle gerbe*.

The following lemma is essentially the complex version of what is called the canonical $\gamma(E)$ -structure in [31].

Lemma 4.11. *The universal cover $\widetilde{P_{SO}}$ of P_{SO} induces in a canonical way a principal $\text{Spin}^c(n) \otimes \widehat{\pi}$ -bundle P over M .*

Proof. It is a direct consequence of lemma 2.33 that $\widetilde{P_{SO}} \rightarrow \widetilde{M}$ is a principal $\text{Spin}(n)$ -bundle. Of course, $\widetilde{P_{SO}} \rightarrow P_{SO}$ is a principal $\pi_1(P_{SO})$ -bundle compatible with the projection to $\widetilde{M} \rightarrow M$. The two group actions on $\widetilde{P_{SO}}$ commute and agree on the diagonal $\mathbb{Z}/2\mathbb{Z}$. Thus, we get an action of $\text{Spin}^c(n) \otimes \widehat{\pi}$ on $P = (\widetilde{P_{SO}} \times S^1)/(\mathbb{Z}/2\mathbb{Z})$. So, we have a principal $\text{Spin}^c(n)$ -bundle $P \rightarrow \widetilde{M}$, a principal π -bundle $\widetilde{M} \rightarrow M$ and an action of $\text{Spin}^c(n) \otimes \widehat{\pi}$ on the total space of the first that is compatible with the two other actions in the sense of lemma 2.32 for the sequence (25). \square

Corollary 4.12. *If L_{spin} denotes the spinor bundle gerbe from example 2.17 then we have $dd(L_{\widehat{\pi}}) = -dd(L_{\text{spin}}) = W_3(M) \in H^3(M, \mathbb{Z})$ with a canonical flat trivialization Q_{spin} .*

Proof. We need to find a trivialization for $L_{\text{spin}} \boxtimes L_{\widehat{\pi}}$, but the latter is the lifting bundle gerbe for the extension

$$1 \rightarrow S^1 \rightarrow \text{Spin}^c(n) \otimes \widehat{\pi} \rightarrow SO(n) \times \pi \rightarrow 1$$

and the principal $SO(n) \times \pi$ -bundle $P_{SO} \times_M \widetilde{M}$. As we have seen in lemma 4.11 the universal cover induces a principal $\text{Spin}^c(n) \otimes \widehat{\pi}$ -bundle P over M covering the bundle $P_{SO} \times_M \widetilde{M}$. Therefore $dd(L_{\text{spin}}) + dd(L_{\widehat{\pi}}) = dd(L_{\text{spin}} \boxtimes L_{\widehat{\pi}}) = 0$. Since the principal $U(1)$ -bundle $P \rightarrow P_{SO} \times_M \widetilde{M}$ reduces to the covering $\widetilde{P_{SO}} \rightarrow P_{SO} \times_M \widetilde{M}$, the associated line bundle Q_{spin} is flat. \square

Remark 4.13. Since $W_3(M) \in H^3(M, \mathbb{Z})$ is a $\mathbb{Z}/2\mathbb{Z}$ -torsion class, its sign plays no role and we have $W_3(M) = -W_3(M)$.

We are now in the position to define the analogue of the Mishchenko-Fomenko line bundle in the twisted case:

Definition 4.14. Let $c_{\widehat{\pi}} \in H_{\text{gr}}^2(\pi, S^1)$ be the group cocycle of the extension $1 \rightarrow S^1 \rightarrow \widehat{\pi} \rightarrow \pi \rightarrow 1$. The canonical twisted Hilbert A -module bundle for $A = C_{\text{max}}^*(\pi, c_{\widehat{\pi}})$ given by $\mathcal{V}_{\text{max}} = \widetilde{M} \times C_{\text{max}}^*(\pi, c_{\widehat{\pi}})$ is called the (maximal) twisted Mishchenko-Fomenko bundle, likewise we could define a reduced version of this via $\mathcal{V}_{\text{red}} = \widetilde{M} \times_{C_r^*}(\pi, c_{\widehat{\pi}})$.

To summarize what we gained from theorem 4.4 and definition 4.14 we state:

Theorem 4.15. *Let M be an oriented even-dimensional manifold with $\dim(M) \geq 3$, such that the universal cover \widetilde{M} is spin. Then the index of the positive part D_+ of the $\mathbb{Z}/2\mathbb{Z}$ -graded projective Dirac operator D twisted with the twisted Mishchenko-Fomenko bundle satisfies*

$$\theta^{\text{max}}(M) = \text{ind}(D_+^{\mathcal{V}_{\text{max}}, Q_{\text{spin}}}) = \int_M \widehat{A}(M) \text{ch}_{Q_{\text{spin}}}(\mathcal{V}_{\text{max}}) \in K_0(C_{\text{max}}^*(\pi, c_{\widehat{\pi}})) \otimes \mathbb{R}$$

and is an obstruction against the existence of a positive scalar curvature metric on M .

5. ENLARGEABLE MANIFOLDS WITH SPIN UNIVERSAL COVER

In this section we will show how the above machinery can be applied to prove results about index obstructions for non-spin manifolds that still admit a spin structure on the universal cover \widetilde{M} . Since \widetilde{M} may be non-compact, we have to work in a twistedly equivariant setting. As an example we will extend the result of [13, 14] about the Rosenberg index obstruction for enlargeable spin manifolds to enlargeable orientable manifolds with spin structure on the universal cover. Thus, our central definition will be

Definition 5.1. A connected closed oriented manifold M with fixed metric g is called *enlargeable* if the following holds: For every $\varepsilon > 0$, there is a connected cover $\bar{M} \rightarrow M$ carrying a spin structure and an ε -contracting map

$$\bar{\psi}_\varepsilon: (\bar{M}, \bar{g}) \rightarrow (S^n, g_0)$$

which is constant outside a compact subset of \bar{M} and of nonzero degree. Here, \bar{g} denotes the induced metric on \bar{M} and g_0 is the standard metric on S^n .

5.1. Almost flat twisted bundles. Just as in [13, 14] the non-vanishing result will be based on the construction of *almost flat* bundles. These can be assembled to form a flat countertwisting bundle for the projective Dirac operator. This operation represents a homomorphism of K_0 -groups detecting $\theta^{\max}(M)$. We will first describe the definition and the construction of almost flat twisted bundles. The whole machinery in the previous chapters was developed in such a way that the proof of the non-vanishing of $\theta^{\max}(M)$ for orientable enlargeable M with spin universal cover is then very close to the result of Hanke and Schick. We will nevertheless outline all points that have to be altered to match our setting.

Definition 5.2. Let $L_{\bar{\pi}}$ be the Mishchenko-Fomenko bundle gerbe. A sequence $E_i \rightarrow \bar{M}$, $i \in \mathbb{N}$ of smooth bundle gerbe modules for $L_{\bar{\pi}}$ equipped with connections ∇^i will be called a *sequence of almost flat twisted bundles*, if

$$\lim_{i \rightarrow \infty} \|\Omega_i\| = 0,$$

where Ω_i is the curvature of the connection ∇^i and the norm on $\Omega^2(M, \text{end}(E_i))$ is induced by the natural pointwise norm on $\text{end}(E_i) \rightarrow M$ and the maximum norm on the unit sphere bundle in $\Lambda^2(M)$. Moreover, the twistings $\gamma_i^g: E_i \rightarrow g^*E_i$ considered as sections $C(\bar{M}, \text{Hom}(E_i, g^*E_i))$ should be locally Lipschitz continuous with a global Lipschitz constant C independent of i . This means that for each point $\tilde{m} \in \bar{M}$ there exists an open neighborhood U of \tilde{m} , such that E_i and g^*E_i are trivial over U , such that we can view the restriction of γ_i^g to U as an element in $C(U, U(V))$ with V being the typical fiber and we have

$$\|\gamma_i^g(\tilde{m}_1) - \gamma_i^g(\tilde{m}_2)\| \leq C d(\tilde{m}_1, \tilde{m}_2)$$

where the metric is induced by the Riemannian structure pulled back to \bar{M} . Observe that the norm on the left hand side is independent of the choice of trivializations.

Lemma 5.3. Let M be an orientable smooth manifold and let \bar{M} be a cover of M equipped with a spin structure, set $\pi = \pi_1(M)$ and $\bar{\pi} = \pi_1(\bar{M}) \subset \pi$. The Hilbert space $H = \ell^2(\pi/\bar{\pi})$ is in a canonical way a projective representation of π with respect to a cocycle that is cohomologous to $c_{\bar{\pi}}$ given in definition 4.14.

Proof. Let $P = P_{SO}$ be the oriented frame bundle of M and let $\pi: \bar{M} \rightarrow M$ be the covering projection, then $\pi^*P = P \times_M \bar{M}$ is the frame bundle for \bar{M} with respect to the pullback metric. Since \bar{M} carries a spin structure, we get a homomorphism $\tau: \bar{\pi} \rightarrow \pi_1(P) \rightarrow \hat{\pi}$ by example 2.37. This means that the cocycle $c_{\hat{\pi}}$ is cohomologous to one that is trivial on $\bar{\pi} \times \bar{\pi}$. But we can enhance this result a little bit: Choose representatives $a_i \in \pi$, $i \in I$ for the cosets in $\pi/\bar{\pi}$ (with $a_0 = 1$ for $\bar{\pi} \subset \pi$) and lifts $\hat{a}_i \in \hat{\pi}$ of these (with $\hat{a}_0 = \hat{1} = 1$). Let $x \in \pi$, then $x = a_i y$ uniquely for an element $y \in \bar{\pi}$. Set $\hat{x} = \hat{a}_i \tau(y)$, which is a lift of x to $\hat{\pi}$. In particular, for $x a_j = a_k y'$ with $y' \in \bar{\pi}$ we have $\widehat{x a_j} = \hat{a}_k \tau(y')$ and thus

$$\widehat{x a_j y} = \hat{a}_k \tau(y' y) = \hat{a}_k \tau(y') \tau(y) = \widehat{x a_j} \tau(y) .$$

Now define $c'_{\hat{\pi}}: \pi \times \pi \rightarrow S^1$ via

$$c'_{\hat{\pi}}(x, a_j y) = \hat{x} \hat{a}_j \tau(y) \widehat{x a_j y}^{-1} = \hat{x} \hat{a}_j \tau(y) \tau(y)^{-1} \widehat{x a_j}^{-1} = \hat{x} \hat{a}_j \widehat{x a_j}^{-1}$$

for $x \in \pi$ and $y \in \bar{\pi}$. Since $c'_{\hat{\pi}}$ represents the same extension as $c_{\hat{\pi}}$ both cocycles are cohomologous, but the value of $c'_{\hat{\pi}}$ does not depend on the choice of $y \in \bar{\pi}$. Thus, we define the action of π on H by

$$x[v] = c'_{\hat{\pi}}(x, v)[xv]$$

for $x \in \pi$ and $v \in \pi/\bar{\pi}$. This yields a projective unitary representation of π on H with respect to the cocycle $c'_{\hat{\pi}}$. \square

Theorem 5.4. *Let M be an even-dimensional orientable manifold that is enlargeable in the sense of definition 5.1. Let $i \in \mathbb{N}$ be a positive natural number. Then there is a C^* -algebra C_i (which will be constructed in the proof) and a twisted Hilbert C_i -module bundle $E_i \rightarrow \bar{M}$ for the Mishchenko-Fomenko bundle gerbe $L_{\hat{\pi}}$ together with a twisted connection ∇_i that has the following properties: The curvature Ω_i of E_i satisfies*

$$\|\Omega_i\| \leq \frac{1}{i} C$$

where C is a constant depending only on $\dim(M)$. Moreover, there is a split extension

$$0 \rightarrow \mathbb{K} \rightarrow C_i \rightarrow X_i \rightarrow 0$$

with a certain C^* -algebra X_i . In particular, each $K_0(C_i)$ splits off a $\mathbb{Z} = K_0(\mathbb{K})$ summand and the $K_0(\mathbb{K})$ -part of the index of the projective Dirac operator $D_+^{E_i, Q_{\text{spin}}}$ is different from 0.

Proof. Let $2n = \dim(M)$ and $\pi = \pi_1(M)$. Since the Chern character is rationally an isomorphism, there is a vector bundle $F \rightarrow S^{2n}$ with non-vanishing top Chern class $c_n(F) \neq 0$. Choose a connection η_F on F and fix $i \in \mathbb{N}$. Since M is enlargeable, there exists a spin covering $\bar{M} \rightarrow M$ together with a $\frac{1}{i}$ -contracting map

$$\psi: \bar{M} \rightarrow S^{2n} ,$$

which is constant outside a compact subset K of \bar{M} . Let P_F be the principal $U(n)$ -bundle of frames in F . Since ψ is constant on $M \setminus K$ we can choose a trivialization for the principal $U(n)$ -bundle ψ^*P_F over this set:

$$(\psi^*P_F)|_{M \setminus K} \cong (M \setminus K) \times U(n)$$

such that the pullback connection $\psi^*\eta_F$ is flat. Let $\rho: \bar{M} \rightarrow M$ be the covering map, $\tilde{\rho}: \tilde{M} \rightarrow M$ the universal cover and $\bar{\pi} = \pi_1(\bar{M})$. Just like in the proof

of [14, Proposition 1.5] we can cover M by open sets $U_j, j \in I$, such that each component $V_{\lambda,j} \subset \bar{M}$ of $\rho^{-1}(U_j)$ maps diffeomorphically onto U_j , intersects only one component $V_{\mu,k}$ of $\rho^{-1}(U_k)$ for any k and such that ψ^*P_F trivializes over each $V_{\lambda,j}$. Let $J_j = \pi_0(\rho^{-1}(U_j))$ be the index set labeling the components, likewise set $\tilde{J}_j = \pi_0(\tilde{\rho}^{-1}(U_j))$. Let $\tilde{\varphi}_{\alpha,j}: \tilde{J}_j \rightarrow \pi/\bar{\pi}$ be the map that sends αg to $[g] \in \pi/\bar{\pi}$ for $g \in \pi$, where π acts on \tilde{J}_j by deck transformations. Since $\bar{M} = \tilde{M}/\bar{\pi}$, this induces bijections $\varphi_{\lambda,j}: J_j \rightarrow \pi/\bar{\pi}$ for each $\lambda \in J_j$. Note that

$$(26) \quad \varphi_{[\alpha g],j} = g^{-1} \cdot \varphi_{[\alpha],j} .$$

Moreover, if $\lambda \in J_j$ and $\mu \in J_k$ belong to components with non-empty intersection, then $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau)$ if τ and κ intersect.

Now consider the Hilbert space

$$H = \ell^2(\pi/\bar{\pi}) \otimes \mathbb{C}^n .$$

Like in [14] we would like to define two C^* -algebras C_S and C_T to obtain C_i from them, but we have to take the twisting into account. Therefore, we define $C_S \subset \mathcal{B}(H)$ to be the C^* -algebra generated by the group of all permutations of $\pi/\bar{\pi}$ and all multiplications by functions $f: \pi/\bar{\pi} \rightarrow S^1$. So we have permutation operators with S^1 -entries instead of just 1s as a generating set of C_S . Let $C_T \subset \mathcal{B}(H)$ be C^* -algebra generated by linear transformations, which are of the form

$$T: H \rightarrow H \quad ; \quad T([g] \otimes v) = [h] \otimes T'v \quad \text{and} \quad T|_{([g] \otimes \mathbb{C}^n)^\perp} = 0 .$$

for some matrix $T' \in M_n(\mathbb{C})$ and $[g], [h] \in \pi/\bar{\pi}$. Let $C_{S,T}$ be the C^* -algebra generated by C_S and C_T inside of $\mathcal{B}(H)$ and note that C_T yields a 2-sided ideal in $C_{S,T}$. Moreover, C_T is isomorphic either to the compact operators or to a matrix algebra. Applying the stabilization trick of [14] we can without loss of generality assume that the former is the case. Now set

$$C_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in C_T\}$$

just like in [14, Proposition 1.5]. This algebra fits into a split exact sequence

$$0 \rightarrow C_T \rightarrow C_i \rightarrow C_{S,T} \rightarrow 0$$

with the splitting induced by the diagonal map, $C_T \rightarrow C_i$ via $a \mapsto (a, 0)$ and $C_i \rightarrow C_{S,T}$ via $(a, b) \mapsto b$.

We choose trivializations of ψ^*P_F over the sets $V_{\lambda,j} \subset \bar{M}$, where we take the trivialization fixed above if $V_{\lambda,j}$ is a subset of $\bar{M} \setminus K$. This way we get a cocycle on the double intersections $V_{(\lambda,\mu),(j,k)} = V_{\lambda,j} \cap V_{\mu,k}$:

$$T'_{(\lambda,\mu),(j,k)}: V_{(\lambda,\mu),(j,k)} \rightarrow U(n) .$$

We can extend $T'_{(\lambda,\mu),(j,k)}$ to a cocycle with values in the unitary group $U(C_{S,T})$ as follows:

$$T^1_{(\lambda,\mu),(j,k)}(x)(\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\mu,k}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(\bar{x})(v) ,$$

where $\tau \in \pi/\bar{\pi}$ is the index of the component of $\rho^{-1}(U_k)$ that intersects $V_{\kappa,j}$ and \bar{x} denotes the lift of $\rho(x)$ to the component $V_{\kappa,j}$. This map actually does nothing to the first tensor factor by our previous considerations. Let $T^2_{(\lambda,\mu),(j,k)}$ be the constant map with value $1 \in U(C_{S,T})$. $T'_{(\kappa,\tau),(j,k)}$ is different from the identity only for finitely many pairs (κ, τ) . Thus,

$$T_{(\lambda,\mu),(j,k)}: V_{(\lambda,\mu),(j,k)} \rightarrow U(C_i) \quad ; \quad T_{(\lambda,\mu),(j,k)} = (T^1_{(\lambda,\mu),(j,k)}, T^2_{(\lambda,\mu),(j,k)}) .$$

is a well-defined cocycle with values in $U(C_i)$. We therefore get a smooth Hilbert C_i -module bundle $\bar{E}_i \rightarrow \bar{M}$, whose pullback to \widetilde{M} will be $E_i = \widetilde{M} \times_M \bar{E}_i$. By lemma 5.3, the space $\ell^2(\pi/\bar{\pi})$ carries a projective unitary representation of π with respect to the cocycle $c_{\bar{\pi}}$, which induces a projective representation $r: \pi \rightarrow U(C_i)$. For $\alpha, \beta \in \tilde{J}_j$ denote the corresponding components of $\tilde{\rho}^{-1}(U_j)$ by $W_{\alpha,j}$ and $W_{\beta,j}$ respectively. Let $\lambda = [\alpha]$, $\mu = [\beta]$, $\lambda' = [\alpha g^{-1}]$ and $\mu' = [\beta g^{-1}] \in J_j$. We define

$$\gamma^g: W_{\alpha,j} \times C_i \rightarrow W_{\alpha g^{-1},j} \times C_i$$

by left multiplication with $r(g)$. Due to equation (26) and with $\varphi_{\lambda,j}(\kappa) = \varphi_{\mu,k}(\tau) = [h] \in \pi/\bar{\pi}$ we have

$$\begin{aligned} (T_{(\lambda',\mu'),(j,k)}(x) \cdot r(g)) (\varphi_{\lambda,j}(\kappa) \otimes v) &= c_{\bar{\pi}}(g, h) \varphi_{\mu',j}(\tau) \otimes T'_{(\kappa,\tau),(j,k)}(v) \\ &= (r(g) \cdot T_{(\lambda,\mu),(j,k)}(x)) (\varphi_{\lambda,j}(\kappa) \otimes v) . \end{aligned}$$

Thus, γ^g intertwines the transition functions of E_i and $g^*(E_i)$. Therefore it yields a well-defined twisting map

$$\gamma^g: E_i \rightarrow g^*(E_i) .$$

This clearly satisfies the Lipschitz condition, since it even is locally constant.

Let $\eta_{\kappa,j} \in \Omega^1(V_{\kappa,j}, \mathfrak{u}(n))$ be the pullback of η_F via the trivialization. These induce forms in $\Omega^1(V_{\lambda,j}, C_{S,T}^a)$, where $C_{S,T}^a$ denotes the anti-selfadjoint operators in $C_{S,T}$, via

$$\left(\eta_{\lambda,j}^{E_i} \right)_x (\xi) (\varphi_{\lambda,j}(\kappa) \otimes v) = \varphi_{\lambda,j}(\kappa) \otimes (\eta_{\kappa,j})_x (\xi) \cdot v .$$

Since $\eta_{\kappa,j}$ is non-zero only for finitely many κ , we can extend $\eta_{\lambda,j}^{E_i}$ to a well-defined 1-form with values in the anti-selfadjoint operators of C_i by setting it to zero in the second component. These 1-forms inherit their transformation behaviour from the forms $\eta_{\kappa,j}$. Thus, they yield a C_i -linear connection ∇^i on sections of E_i . Just like above it follows from (26) that ∇^i is a twisted connection. Since the norm of the curvature Ω_i of ∇^i coincides with that of $\psi^*\Omega_F$, we have

$$\|\Omega_i\| = \|\psi^*\Omega_F\| \leq \frac{1}{i} C .$$

It remains to be shown that the $K_0(\mathbb{K})$ -part of $\text{ind}(D_+^{E_i, Q_{\text{spin}}})$ does not vanish. Here we proceed exactly as in [14]: Let $\mathcal{T} \subset C_T \cong \mathbb{K}$ be the trace class ideal and let D_i be the algebra given by

$$D_i = \{(c_1, c_2) \in C_{S,T} \times C_{S,T} \mid c_1 - c_2 \in \mathcal{T}\} .$$

Since the proof of lemma [14, lemma 2.4] applies to D_i with the changed $C_{S,T}$ as well, D_i is a unital local C^* -algebra with a trace $\tau(c_1, c_2) = \text{tr}(c_1 - c_2)$, which coincides with the trace of the element after projecting it from C_i to \mathcal{T} . Its C^* -completion is C_i . Since $K_0(C_i) \cong K_0(D_i)$, we can extend \dim_{τ} from definition 3.24 to a functional on $K_0(C_i)$ and it suffices to prove that $\dim_{\tau}(\text{ind}(D_+^{E_i, Q_{\text{spin}}})) \neq 0$. The transition functions in the definition of E_i actually take values in $U(D_i)$ and thus lead to a twisted D_i -module bundle \mathcal{E}_i in the sense of section 3.1.3. By

theorem 4.4, equation (20) and the fact that $\text{ch}(Q_{\text{spin}}) = 1$ we have

$$\begin{aligned} \dim_{\tau}(\text{ind}(D_+^{E_i, Q_{\text{spin}}})) &= \int_M \widehat{A}(M) \dim_{\tau}(\text{ch}_{Q_{\text{spin}}}(E_i)) \\ &= \int_M \widehat{A}(M) \dim_{\tau}(\text{ch}_{Q_{\text{spin}}}(\mathcal{E}_i)) = \int_M \widehat{A}(M) \text{ch}_{\tau}(\mathcal{E}_i) . \end{aligned}$$

We can identify $\Omega_{\mathcal{E}_i} \in \Omega^2(M, \text{end}(\mathcal{E}_i))$ with an equivariant form in $\Omega^2(\widetilde{M}, \text{End}(\mathcal{E}_i))$. If we carry out the integration over a single subset $U_j \subset M$, we could integrate instead over the subset $W_{\alpha, j} \subset \widetilde{M}$ for some $\alpha \in \widetilde{J}_j$. This is independent of the choice of α by equivariance. But over $W_{\alpha, j}$ the form $\tau(\Omega_{\mathcal{E}_i} \wedge \cdots \wedge \Omega_{\mathcal{E}_i})|_{W_{\alpha, j}} \in \Omega^{\text{even}}(W_{\alpha, j}, \mathbb{R})$ coincides with the sum of all $\Omega_{\psi^* F} \wedge \cdots \wedge \Omega_{\psi^* F}|_{V_{\kappa, j}} \in \Omega^{\text{even}}(V_{\kappa, j}, \mathbb{R})$ over $\kappa \in J_j$ by the definition of the trace. Using a partition of unity on M we see that

$$\int_M \widehat{A}(M) \text{ch}_{\tau}(\mathcal{E}_i) = \int_{\widetilde{M}} \widehat{A}(\widetilde{M}) \text{ch}(\psi^* F - \underline{\mathbb{C}}^n) ,$$

Since the class of $\text{ch}(\psi^* F - \underline{\mathbb{C}}^n)$ is concentrated in degree n we see that the above term is non-vanishing. \square

Remark 5.5. Due to the stabilization trick mentioned in the proof the fibers of E_i are isomorphic to $t_i C_i$ for some projection $t_i \in C_i$, where $t_i = 1$ if \widetilde{M} is non-compact.

Having the sequence E_i of almost flat twisted bundles at hand, we can form the C^* -algebra

$$A = \prod_{i \in \mathbb{N}} C_i$$

of bounded sequences with i th entry in C_i , in which the norm closure of the sequences with only finitely many non-zero entries

$$A' = \overline{\bigoplus_{i \in \mathbb{N}} C_i}^{\|\cdot\|}$$

is a two-sided ideal and we set $Q = A/A'$. Let A_i be the ideal in A consisting of sequences that are 0 everywhere, but in the i th entry. We can assemble the bundles E_i into a twisted Hilbert A -module bundle.

Theorem 5.6. *There is a smooth twisted Hilbert A -module bundle $E \rightarrow \widetilde{M}$ together with a twisted connection*

$$\nabla^E : C^\infty(\widetilde{M}, E) \rightarrow C^\infty(\widetilde{M}, T^* \widetilde{M} \otimes E)$$

such that the following holds:

- $E \cdot A_i$ is isomorphic to E_i as a twisted Hilbert C_i -module bundle.
- The connection preserves the subbundles $E \cdot A_i$.
- The sequence of curvatures $\Omega_i \in \Omega^2(M, \text{end}(E \cdot A_i))$ of the connection induced on $E \cdot A_i$ by ∇^E satisfies

$$\lim_{i \rightarrow \infty} \|\Omega_i\| = 0 .$$

Proof. The idea is to see that the product bundle $E_L = \Delta_{\widetilde{M}}^* (\prod_{i \in \mathbb{N}} E_i)$, where

$$\Delta_{\widetilde{M}} : \widetilde{M} \rightarrow \prod_{i \in \mathbb{N}} \widetilde{M}$$

is the diagonal map, has locally Lipschitz continuous transition functions. This parallels the construction given in the proof of [13, lemma 2.1] with the only difference that we have to work equivariantly over \widetilde{M} , so we just sketch the differences and refer to [13] for the details: We cover M by subsets U_j , each of them diffeomorphic to I^n , where $I = [0, 1]$, such that $\widetilde{M} \rightarrow M$ is trivial over U_j via

$$\phi_j: U_j \times \pi \rightarrow \widetilde{M} \Big|_{U_j} .$$

We can find trivializations

$$\psi_{i,j}^1: \phi_j^* E_i \Big|_{U_j \times \{1\}} \rightarrow I^n \times t_i C_i$$

of $\phi_j^* E_i \Big|_{U_j \times \{1\}}$, such that constant sections of $\phi_j^* E_i$ over $I^k \times \{0, \dots, 0\}$ are parallel with respect to ∇_{∂_l} for $1 \leq l \leq k$, where ∇ denotes the connection induced by ∇^{E_i} . Using the twisting we can extend $\psi_{i,j}^1$ to a trivialization $\psi_{i,j}$ of $\phi_j^* E_i \Big|_{U_j \times \pi}$ with components $\psi_{i,j}^g$ with $g \in \pi$. Let $\eta_{i,j}^g \in \Omega^1(I^n, t_i C_i t_i)$ be the pullback of the connection 1-form of ∇^{E_i} . The way the trivializations are constructed is crucial to prove the estimate given in [13, lemma 2.3], which now still holds and we have

$$\|\eta_{i,j}^g\| \leq n \cdot \|\Omega_{i,j}^g\| ,$$

where $\Omega_{i,j}^g$ denotes the curvature of $\eta_{i,j}^g$. The right hand side of the above inequality is independent of $g \in \pi$. Thus, our control of the curvatures carries over to an upper bound on the local connection 1-forms. The trivializations $\psi_{i,j}$ induce transition maps

$$\psi_{i,(j,k)}: (U_j \cap U_k) \times \pi \rightarrow U(t_i C_i t_i)$$

and the upper bound on the local connection 1-forms yields an upper bound on the norm of the derivative $D_{(x,g)} \psi_{i,(j,k)}$ just as described in [13, lemma 2.5, proposition 2.6] proving Lipschitz continuity of the transition functions. The Lipschitz condition on the twisting maps ensures that the product of the γ_i^g is continuous, when considered as an element in $C(\widetilde{M}, \text{Hom}(E_L, g^* E_L))$. Thus, E_L is a continuous twisted Hilbert A -module bundle. Note that $\pi = \pi_1(M)$ acts via the adjoint action unitarily on C_i and we set $\mathcal{A} = \widetilde{M} \times_{\text{Ad}} C_i$. E_L corresponds to a projection $t_L \in C(M, \mathcal{A})$, which we can approximate by a projection in $C^\infty(M, \mathcal{A})$ to obtain a smooth twisted Hilbert A -module bundle E by the construction given in theorem 3.9. We have $E \cdot A_i \cong E_L \cdot A_i = E_i$. The isomorphism may only be continuous, but it can be smoothed.

To construct the connection ∇^E we only need to give local connection 1-forms over the sets $\phi_j(U_j \times \{1\}) \subset \widetilde{M}$ and extend them equivariantly via γ^g to get connection forms over the images of $U_j \times \pi$, which can be patched together with a partition of unity on M . The construction takes the local forms of the E_i and uses a convolution argument to get a smooth form on the product. This is exactly the same as in [13]. \square

The twisting γ^g maps the subbundle $E \cdot A'$ into itself, therefore the quotient $\mathcal{W} = E/(E \cdot A')$ is a smooth twisted Hilbert Q -module bundle equipped with a flat connection ∇^Q and typical fiber tQ for some projection $t \in Q$. If we now fix a basepoint $\tilde{m} \in \widetilde{M}$, we get a projective holonomy representation by our observations in section 4.1

$$(\pi_1(M), c_{\tilde{\pi}}) \rightarrow \text{End}(\mathcal{W}_{\tilde{m}}) = tQt .$$

By the universal property of the maximal twisted group C^* -algebra, this extends to a $*$ -homomorphism

$$\phi: C_{\max}^*(\pi_1(M), c_{\widehat{\pi}}) \rightarrow Q$$

As a consequence of the naturality proven in corollary 3.38, the induced map $\phi_*: K_0(C^*(\pi_1(M), c_{\widehat{\pi}})) \rightarrow K_0(Q)$ maps $\theta^{\max}(M)$ to $\text{ind}(D_+^{\mathcal{W}', Q_{\text{spin}}})$, where

$$\mathcal{W}' = \widetilde{M} \times tQ = \widetilde{M} \times \mathcal{W}_{\widetilde{m}}.$$

But using the parallel transport (see section 4.1) and its equivariance (23) with respect to γ^g we see that \mathcal{W}' is isomorphic to \mathcal{W} as a twisted Hilbert Q -module bundle.

Theorem 5.7. *Let M be a closed compact smooth orientable even-dimensional manifold with $\dim(M) \geq 3$ and \widetilde{M} spin that is enlargeable in the sense of definition 5.1. Then we have*

$$\theta^{\max}(M) \neq 0 \in K_0(C_{\max}^*(\pi_1(M), c_{\widehat{\pi}})).$$

Proof. As we saw above, we have

$$\phi_*(\theta^{\max}(M)) = \text{ind}(D_+^{\mathcal{W}, Q_{\text{spin}}}) \in K_0(Q)$$

By [14], the group $K_0(Q)$ splits off a summand

$$\prod_{i \in \mathbb{N}} K_0(\mathbb{K}) / \bigoplus_{i \in \mathbb{N}} K_0(\mathbb{K}) \cong \prod_{i \in \mathbb{N}} \mathbb{Z} / \bigoplus_{i \in \mathbb{N}} \mathbb{Z}$$

and the image of $\text{ind}(D_+^{\mathcal{W}, Q_{\text{spin}}})$ in the latter group corresponds to the sequence

$$z_i = \left[p_* \left(\text{ind}(D_+^{E \cdot A_i, Q_{\text{spin}}}) \right) \right] = \left[p_* \left(\text{ind}(D_+^{E_i, Q_{\text{spin}}}) \right) \right],$$

where $p: C_i \rightarrow \mathbb{K}$ is the projection. By theorem 5.4 it has only non-vanishing entries. \square

Remark 5.8. Extending the suspension argument from [13] it is easy to drop the assumption about even-dimensionality. Relaxing the condition about the orientability of M requires incorporating orientation twists of K -theory into the setup, which can be seen as a special case of twisted $\mathbb{Z}/2\mathbb{Z}$ -equivariant K -theory as has been observed by Karoubi [17, Remark 6.16], [15]. These can also be described by gerbes (see the Jandl gerbes in [10] and the functor $K_{\pm}(X)$ in [1], which is naturally equivalent with Karoubi's definition), therefore the above argument should generalize to non-orientable manifolds as well. Nevertheless, it seems to be impossible to drop the spin condition for the covers \widetilde{M} in the definition of enlargeability, since our construction of a projective representation with the right cocycle relies on that.

REFERENCES

- [1] Michael Atiyah and Michael Hopkins. A variant of K -theory: K_{\pm} . In *Topology, geometry and quantum field theory*, volume 308 of *London Math. Soc. Lecture Note Ser.*, pages 5–17. Cambridge Univ. Press, Cambridge, 2004. 51
- [2] Michael Atiyah and Graeme Segal. Twisted K -theory. *Ukr. Mat. Visn.*, 1(3):287–330, 2004. 1, 23
- [3] Michael Francis Atiyah. *Elliptic operators and compact groups*. Lecture Notes in Mathematics, Vol. 401. Springer-Verlag, Berlin, 1974. 31

- [4] Saad Baaĵ and Pierre Julg. Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules hilbertiens. *C. R. Acad. Sci. Paris Sér. I Math.*, 296(21):875–878, 1983. [33](#)
- [5] Bruce Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998. [29](#), [33](#)
- [6] Peter Bouwknegt, Alan L. Carey, Varghese Mathai, Michael K. Murray, and Danny Stevenson. Twisted K -theory and K -theory of bundle gerbes. *Comm. Math. Phys.*, 228(1):17–45, 2002. [1](#), [8](#), [9](#), [10](#), [24](#)
- [7] Ulrich Bunke. A K -theoretic relative index theorem and Callias-type Dirac operators. *Math. Ann.*, 303(2):241–279, 1995. [33](#), [36](#)
- [8] Alan L. Carey and Bai-Ling Wang. Thom isomorphism and push-forward map in twisted K -theory. *J. K-Theory*, 1(2):357–393, 2008. [2](#), [8](#), [9](#)
- [9] P. Donovan and M. Karoubi. Graded Brauer groups and K -theory with local coefficients. *Inst. Hautes Études Sci. Publ. Math.*, (38):5–25, 1970. [1](#), [2](#), [24](#)
- [10] Jürgen Fuchs, Thomas Nikolaus, Christoph Schweigert, and Konrad Waldorf. Bundle gerbes and surface holonomy. In *European Congress of Mathematics*, pages 167–195. Eur. Math. Soc., Zürich, 2010. [51](#)
- [11] Kiyonori Gomi. Connections and curvings on lifting bundle gerbes. *J. London Math. Soc. (2)*, 67(2):510–526, 2003. [5](#), [6](#)
- [12] Mikhael Gromov and H. Blaine Lawson, Jr. Spin and scalar curvature in the presence of a fundamental group. I. *Ann. of Math. (2)*, 111(2):209–230, 1980. [2](#)
- [13] Bernhard Hanke and Thomas Schick. Enlargeability and index theory. *J. Differential Geom.*, 74(2):293–320, 2006. [2](#), [36](#), [45](#), [50](#), [51](#)
- [14] Bernhard Hanke and Thomas Schick. Enlargeability and index theory: infinite covers. *K-Theory*, 38(1):23–33, 2007. [2](#), [45](#), [47](#), [48](#), [51](#)
- [15] Max Karoubi. Algèbres de Clifford et K -théorie. *Ann. Sci. École Norm. Sup. (4)*, 1:161–270, 1968. [51](#)
- [16] Max Karoubi. *K-theory*. Springer-Verlag, Berlin, 1978. An introduction, Grundlehren der Mathematischen Wissenschaften, Band 226. [26](#)
- [17] Max Karoubi. Twisted K -theory—old and new. In *K-theory and noncommutative geometry*, EMS Ser. Congr. Rep., pages 117–149. Eur. Math. Soc., Zürich, 2008. [1](#), [51](#)
- [18] E. C. Lance. *Hilbert C^* -modules*, volume 210 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists. [16](#), [32](#)
- [19] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989. [39](#), [40](#)
- [20] V. Mathai, R. B. Melrose, and I. M. Singer. Fractional analytic index. *J. Differential Geom.*, 74(2):265–292, 2006. [9](#), [10](#), [30](#)
- [21] V. Mathai, R. B. Melrose, and I. M. Singer. Equivariant and fractional index of projective elliptic operators. *J. Differential Geom.*, 78(3):465–473, 2008. [31](#)
- [22] M. K. Murray. Bundle gerbes. *J. London Math. Soc. (2)*, 54(2):403–416, 1996. [1](#), [3](#), [5](#), [6](#), [7](#), [8](#)
- [23] M. K. Murray. An Introduction to Bundle Gerbes. *arXiv:0712.1651*, 2008. [3](#)

- [24] Michael K. Murray and Michael A. Singer. Gerbes, Clifford modules and the index theorem. *Ann. Global Anal. Geom.*, 26(4):355–367, 2004. [2](#), [9](#), [32](#)
- [25] Karl-Hermann Neeb. Central extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)*, 52(5):1365–1442, 2002. [7](#)
- [26] Ulrich Pennig. Twisted K -theory with coefficients in C^* -algebras. *arXiv:1103.4096v1 [math.KT]*, 2011. [2](#)
- [27] Jonathan Rosenberg. C^* -algebras, positive scalar curvature, and the Novikov conjecture. III. *Topology*, 25(3):319–336, 1986. [1](#)
- [28] Jonathan Rosenberg. Manifolds of positive scalar curvature: a progress report. In *Surveys in differential geometry. Vol. XI*, volume 11 of *Surv. Differ. Geom.*, pages 259–294. Int. Press, Somerville, MA, 2007. [1](#)
- [29] Thomas Schick. A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture. *Topology*, 37(6):1165–1168, 1998. [1](#)
- [30] Thomas Schick. L^2 -index theorems, KK -theory, and connections. *New York J. Math.*, 11:387–443 (electronic), 2005. [20](#), [27](#), [29](#), [33](#), [36](#)
- [31] Stephan Stolz. Concordance classes of positive scalar curvature metrics. *preprint*. [1](#), [16](#), [37](#), [43](#)
- [32] Stephan Stolz. Simply connected manifolds of positive scalar curvature. *Ann. of Math. (2)*, 136(3):511–540, 1992. [1](#)
- [33] Atsushi Tomoda. On the splitting principle of bundle gerbe modules. *Osaka J. Math.*, 44(1):231–246, 2007. [24](#)
- [34] Stéphane Vassout. Unbounded pseudodifferential calculus on Lie groupoids. *J. Funct. Anal.*, 236(1):161–200, 2006. [32](#)

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